1. (NH) Define sub-$G$-sets. See that orbits are sub-$G$-sets.

(b) Given a family of $G$-sets $(X_i)_{i \in I}$, define the $G$-set $\bigsqcup_{i \in I} X_i$ (disjoint union).

(c) Notice that any $G$-set $X$ is canonically isomorphic to the $G$-set which is the disjoint union of the orbits of $X$.

2. (a) Let $G$ be a group and $H \subset G$ a subgroup. Show that for a $G$-set $X$, the set of homomorphisms of $G$-sets between $G/H$ and $X$ is in bijection with the set of elements in $X$ whose stabiliser contains $H$.

(b) Let $G$ be a group and $H \subset G$ a subgroup. Describe the group of automorphisms of the $G$-set $G/H$ (so, $G/H$ is a $G$-set, we consider all its automorphisms as a $G$-set, and the set of all those is a group - we want to describe it in concretely). Hint: the answer is that this group of automorphisms is isomorphic to the group $(N_G(H)/H)^{op}$. You should construct the isomorphism (and proof that you indeed get an isomorphism).

3. (a) Write explicitly the elements of the centraliser of $\sigma = (12)(34)(567) \in S_7$, i.e. the subgroup $C \subset S_7$ consisting of elements commuting with $\sigma$. Hint: Think about $\tau \sigma \tau^{-1} = (\tau(1)\tau(2))(\tau(3)\tau(4))(\tau(5)\tau(6)\tau(7))$. 

(NH) stands for "Not for Handing in"; The exercises with this mark are just for extra exercise, not for grade.
(b) Verify the class formula for $S_4$: In other words, describe representatives for conjugacy classes, calculate the number of elements in their centralisers, and then plug into the formula and see that you get a true statement.

4. Let $p$ be a prime. Prove that any group with $p^2$ elements is either isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ or to $\mathbb{Z}/p^2\mathbb{Z}$.

5. Let $p < q$ be primes. We saw that if $p$ does not divide $q - 1$, then any group with $pq$ elements is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$. Show that if $p | q - 1$, there are, up to isomorphism, exactly two groups of order $pq$ (we more or less showed how to do that in class).

6. (NH) Let $1 \to K \to G \to H \to 1$ be a short exact sequence. Show that if $\alpha$ has a left inverse, then $\beta$ has a right inverse. Give an example when $\beta$ has a right inverse but $\alpha$ has no left inverse (of course, by "inverse" we mean an inverse homomorphism).

7. (NH)
   
   (a) Let $1 \to K \to G \to H \to 1$ be a short exact sequence. Suppose that $K$ is abelian. Construct an action of $H$ on $K$ (Hint: lift, then conjugate; Show that result does not depend on lift).

   (b) Given a short exact sequence $1 \to K \to G \to H \to 1$, we say that it is a central extension of $H$ by $K$ (or that $G$ is a central extension of $H$ by $K'$), if $K$ lies in the center of $G$. Show that this is equivalent to the action of the previous item being trivial.

8. (NH) Let $H, K$ be groups, with $K$ abelian.

   (a) Let $1 \to K \to G \to H \to 1$ be a short exact sequence. Choose a set-theoretic section $\ell$ to $\beta$ - i.e. a function $\ell : H \to G$ s.t. $\beta \circ \ell = id_H$ but $\ell$ is not necessarily a homomorphism. Notice that such an $\ell$ always exist. Show that such an $\ell$ gives us an identification of $X$ with $K \times H$ as sets - via $(k, h) \mapsto \alpha(k) \ell(h)$. Notice that choosing such an $\ell$ is the same as choosing a set of representatives for the $K$ cosets in $G$. 

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(b) Let $1 \to K \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 1$ be a central extension of $H$ by $K$, and choose a set-theoretic section $\ell$ as above. For $x, y \in H$, denote by $c(x, y) \in K$ the unique element satisfying $\ell(x)\ell(y) = \alpha(c(x, y)) \cdot \ell(xy)$ (understand that indeed there is a unique such element). Show that the following is satisfied:

$$c(y, z) \cdot c(x, yz) = c(x, y) \cdot c(xy, z)$$

for all $x, y, z \in H$. Show that this implies that $c(x, e) = c(e, z)$ for all $x, z \in H$

(c) Show that if we choose some other section $\ell'$ (and get a corresponding $c'$), there exists a function $b : H \to K$ such that

$$\frac{c'(x, y)}{c(x, y)} = \frac{b(x)b(y)}{b(xy)}$$

for all $x, y \in H$.

(d) Let us be given a function $c : H \times H \to K$, satisfying the relation:

$$c(y, z) \cdot c(x, yz) = c(x, y) \cdot c(xy, z)$$

for all $x, y, z \in H$. Show that we can define a group structure on $K \times H$ as follows: $(c, x)(d, y) = (cd \cdot c(x, y), xy)$. Call this new group, say, $(K, H)_c$. Show that we obtain a central extension of $H$ by $K$:

$$1 \to K \to (K, H)_c \to H \to 1.$$ 

(e) Let $Z^2(H, K)$ be the set of maps $c : H \times H \to K$ satisfying the relation

$$c(y, z) \cdot c(x, yz) = c(x, y) \cdot c(xy, z)$$

for all $x, y, z \in H$. Let $C^1(H, K)$ be the set of all maps $b : H \to K$. Equip $Z^2(H, K)$ and $C^1(H, K)$ with a group structure of just pointwise multiplication in $K$. Consider the map

$$\delta : C^1(H, K) \to Z^2(H, K)$$

given by

$$(\delta b)(x, y) = \frac{b(x)b(y)}{b(xy)}.$$ 

Show that $\delta$ is a well-defined homomorphism. The group $H^2(H, K) : = Z^2(H, K)/\text{Im}(\delta)$ is called the second cohomology.
(f) Consider two central extensions of $H$ by $K$:

\[ 1 \to K \to G \to H \to 1 \]

and

\[ 1 \to K \to G' \to H \to 1. \]

An isomorphism between them, as central extensions of $H$ by $K$, is an isomorphism of them as short exact sequences, but with the left and right isomorphisms being the identity map. From all the above, try to analyze and see that the set of isomorphism classes of central extensions of $H$ by $K$ is in bijection with the group $H^2(H,K)$. 