Homework 03 - Ma 5a, Caltech

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(NH) stands for ”Not for Handing in”; The exercises with this mark are just for extra exercise, not for grade.

1. Let $G$ be a group and $K \subset H \subset G$ subgroups.
   (a) Construct a surjective map $G/K \to G/H$, and an injective map $H/K \to G/K$.
   (b) Show that if $[G : K]$ is finite then so are $[G : H]$ and $[H : K]$.
   (c) Suppose that $[G : K]$ is finite. Show that $[G : K] = [G : H] \cdot [H : K]$.
       Direction: One way could be to choose representatives $g_1, \ldots, g_s$ for left cosets of $H$ in $G$, and representatives $h_1, \ldots, h_r$ for left cosets of $K$ in $H$, and then to show that $g_1h_1, \ldots, g_1h_r, \ldots, g_sh_1, \ldots, g_sh_r$ are representatives for the left cosets of $K$ in $G$.
   (d) (NH) Notice from the above that if $[G : H]$ and $[H : K]$ are finite, then so is $[G : K]$.
   (e) (NH) Notice, that there is no canonical bijection between $G/K$ and $G/H \times H/K$ to evidence the equality of item (c). Rather, one identifies non-canonically the fibers of $G/K \to G/H$ with $H/K$.

2. Let $G$ be a group, and $H \subset G$ a subgroup.
   (a) (NH) Recall that $H$ is called normal in $G$, if $gHg^{-1} = H$ for all $g \in G$.
   (b) Show that $H$ is normal in $G$ if $gHg^{-1} \subset H$ for all $g \in G$. 
(c) Show that if either $|H|$ or $[G : H]$ are finite, and for some $g \in G$ we have $gHg^{-1} \subset H$, then $gHg^{-1} = H$.

(d) (NH) Give an example of a group $G$, a subgroup $H \subset G$ and an element $g \in G$, such that $gHg^{-1} \subset H$ but $gHg^{-1} \neq H$.

3. (a) Let $G$ be a group, and $H \subset G$ a subgroup of index 2. Show that $H$ is normal in $G$. Hint: Think what are the left and right cosets of $H$ in $G$.

(b) Let $(a_1 \ldots a_m) \in S_n$ be a cycle, and $\sigma \in S_n$. Show that

$$\sigma \circ (a_1 \ldots a_m) \circ \sigma^{-1} = (\sigma(a_1) \ldots \sigma(a_m)).$$

(c) Write down all the subgroups of $S_3$, and determine which of them are normal in $S_3$.

4. (a) Let $n \geq 5$. Show that $D_{2n}$ admits at most one normal subgroup of index $n$, and that such a subgroup exists if and only if $n$ is even.

(b) (NH) Let $N \subset D_{2n}$ be the subgroup of the previous item (when $n$ is even). Show that $D_{2n}/N$ is isomorphic, as a group, to $D_n$.

(c) (NH) Remark: I could not for now find a satisfying geometric reason for the isomorphism of the previous item (that would mean making $D_{2n}$ act on the set of vertices of a regular $n/2$-gon, or something like that...).

5. Let $G$ be a group. A subgroup $H \subset G$ is called characteristic, if for every automorphism $\alpha \in \text{Aut}(G)$, one has $\alpha(H) = H$.

(a) Show that a characteristic subgroup of $G$ is a normal subgroup of $G$.

(b) (NH) Give an example of a group $G$ and a normal subgroup of it which is not a characteristic subgroup of it.

(c) Let $K \subset H \subset G$ be subgroups. Suppose that $H$ is normal in $G$, and $K$ is characteristic in $H$. Show that $K$ is normal in $G$.

(d) (NH) Some examples of characteristic subgroups: $G$ itself, $\{e\}$, the center $Z(G) := \{g \in G \mid gh = hg \forall h \in G\}$, if $G$ is abelian then $G_n = \{g \in G \mid g^n = e\}$ for a fixed integer $n$. In general, the idea is that any subgroup which can be described from the ”group structure” of $G”$ without choices”, should be characteristic.
6. (NH)

(a) Show that $S_n$ is generated by the subset of transpositions (cycles of length 2). Hint: Show first that any cycle in $S_n$ can be written as a product of transpositions.

(b) Show that $S_n = \langle (12), (23), \ldots, ((n-1)n) \rangle$. Hint: You might start by showing that a transposition $(ij)$ can be written as a product of transpositions from our list, by induction on $|j - i|$.

(c) Show that $S_n = \langle (12), (12 \cdots n) \rangle$.

7. (NH) Let $G$ be a group.

(a) Let $S \subset G$ be a subset. Show that $\langle S \rangle$ is normal in $G$, if and only if $gs^{-1}g^{-1} \in \langle S \rangle$ for all $g \in G$ and $s \in S$.

(b) Show that if $N_1, N_2 \subset G$ are normal subgroups of $G$, then $N_1 \cap N_2$ is a normal subgroup of $G$.

(c) Show that if $N \subset G$ is a normal subgroup of $G$ and $H \subset G$ is a subgroup of $G$, then $NH := \{nh : n \in N, h \in H\}$ is a subgroup of $G$, and also that $NH = HN$.

(d) Let $K \subset H \subset G$ be subgroups. Show that if $K$ is normal in $G$, then $K$ is normal in $H$.

(e) Give an example of a group $G$ and subgroups $K \subset H \subset G$ such that $K$ is normal in $H$, $H$ is normal in $G$, but $K$ is not normal in $G$.

(f) For a subset $S \subset G$, define $N_G(S) := \{g \in G \mid gsg^{-1} = S\}$ (it is called the normalizer of $S$ in $G$). Show that $N_G(S)$ is a subgroup of $G$. Describe $N_G(S)$ as the stabilizer of $S$ under a relevant group action of $G$ on the set of subsets of $G$.

(g) Let $H \subset G$ be a subgroup. Show that $H \subset N_G(H)$, and that $N_G(H) = G$ if and only if $H$ is normal in $G$.

8. (NH) Let $G$ be a group.

(a) Let me define an abstract quotient of $G$ as a pair $(H, \phi)$ consisting of a group $H$ and a surjective homomorphism $\phi : G \to H$. We say that two abstract quotients of $G$, $(H_1, \phi_1)$ and $(H_2, \phi_2)$, are isomorphic, if there exists an isomorphism of groups $\alpha_{21} : H_1 \to H_2$
such that $\alpha_2 \circ \phi_1 = \phi_2$. Although abstract quotients of $G$ do not form a set (rather, a "class"), isomorphism classes of abstract quotients of $G$ are a perfectly valid set. Understand that from what we did in class, it follows that there is a bijection between the set of isomorphism classes of abstract quotients of $G$, and the set of normal subgroups of $G$.

(b) In a similar fashion, we can define an abstract subobject of $G$ as a pair $(H, \phi)$ consisting of a group $H$ and an injective homomorphism $\phi : H \to G$. Again, one has the notion of an isomorphism of abstract subobjects (understand what it is!). This time, however, the issue is much simpler; The set of isomorphism classes of abstract subobjects of $G$ is in bijection with the set of subgroups of $G$ (where to an abstract subobject $(H, \phi)$ we associate $\text{Im}(\phi)$). Why does it look simpler? Perhaps because our atomistic perception forces us to define sets as collections of elements, so that we have set things up so that subobjects are more on the surface than quotients.

9. (NH) Let $G$ be a group. Let us call an equivalence relation $\sim$ on $G$ groupic, if:

- $g_1 \sim g_2$ and $g_3 \sim g_4$ imply $g_1 g_3 \sim g_2 g_4$.
- $g_1 \sim g_2$ implies $g_1^{-1} \sim g_2^{-1}$.

(In fact, one easily sees that the second condition already follows from the first).

(a) Show that there is a natural bijection, between the set of normal subgroups of $G$ and the set of groupic equivalence relations on $G$. Hint: First construct maps in both directions, as follows: Given a normal subgroup $N \subset G$, define $g_1 \sim g_2$ if $g_2^{-1} g_1 \in N$. Given a groupic equivalence relation $\sim$, define a normal subgroup by $N = \{ g \in G \mid e \sim g \}$. Show that those maps are well-defined and mutually-inverse.

(b) Show that an equivalence relation $\sim$ on $G$ is groupic if and only if:

- $g_1 \sim g_2$ implies $gg_1 \sim gg_2$ and $g_1 g \sim g_2 g$.
(c) Find the correct requirements on an equivalence relation on $G$, such that there will be a natural bijection between the set of such equivalence relations and the set of all subgroups of $G$ (not necessarily normal). Remark: You will find out that there are two such "correct" requirements (related to left-right symmetry) - and the sets of equivalence relations satisfying those are in bijection induced by taking inverse in the group.

10. (NH) Let $V, W$ be vector spaces over a finite field $F$, and $T : V \rightarrow W$ a linear map. We can also regard $V, W$ as abelian groups under addition, and $T$ a homormophism of groups. Explain, in case $V$ is finite-dimensional over $F$, how the formula from linear algebra $\dim(\text{Im}(T)) = \dim(V) - \dim(\ker(T))$ matches with the formula from group theory $|\text{Im}(T)| = \frac{|G|}{|\ker(T)|}$. 