1. Recall that for two positive integers $n, m$, there is a unique positive integer $k = lcm(n, m)$ ("lowest common multiply"), satisfying

$$k | r \iff n | r \text{ and } m | r$$

for all $r \in \mathbb{Z}$. For example, $k = mn$ if and only if $m$ and $n$ are relatively prime.

(a) (NH) Compute $lcm(24, 90)$.

(b) Let $G, H$ be groups, and $g \in G, h \in H$. Consider $(g, h) \in G \times H$. Show that if $o(g) = \infty$ or $o(h) = \infty$, then $o((g, h)) = \infty$. Furthermore, show that if $o(g) \neq \infty$ and $o(h) \neq \infty$, then $o((g, h)) = lcm(o(g), o(h))$.

(c) Let $G, H$ be groups, and $\phi \in Hom(G, H)$. Show that for every $g \in G$, one has $o(\phi(g)) | o(g)$. Furthermore, show that if $\phi$ is a monomorphism, then $o(\phi(g)) = o(g)$ for every $g \in G$.

(d) Let $G$ be a group, and $g, h \in G$ two commuting elements, i.e. $gh = hg$. Show that $o(gh) | lcm(o(g), o(h))$.

(e) (NH) Find a group $G$ and two elements $g, h \in G$, such that $o(g) = 2, o(h) = 2, o(gh) = \infty$ (contrasting the previous item).

(f) Let $G, H$ be groups such that $|G|$ is relatively prime to $|H|$. Show that $|Hom(G, H)| = 1$. 

1
2. Recall that we defined cycles \((a_1 \cdots a_m) \in S_n\) (we call \(m\) the length of that cycle).

(a) Calculate the order of a cycle of length \(m\).

(b) How many cycles of length \(n\) are there in \(S_n\) (provide an explanation of your answer)?

(c) \((\text{NH})\) How many cycles of length 2 are there in \(S_n\) (provide an explanation of your answer)?

(d) \((\text{NH})\) Let \(\sigma \in S_n\) and let \(\sigma = \tau_1 \cdots \tau_r\) be it’s cyclic decomposition. Show that \(o(\sigma) = \text{lcm}(o(\tau_1), \ldots, o(\tau_r))\).

(e) Write the cycle decomposition of

\[
\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 1 & 3 & 10 & 7 & 9 & 5 & 6 & 8 & 2 \end{pmatrix} \in S_{10},
\]

and calculate \(o(\sigma)\) (you can use item (d) for that).

3. Recall that given a subgroup \(H \subset G\), a left coset of \(H\) is a subset of \(G\) of the form \(gH = \{gh : h \in H\}\) for some \(g \in G\) (and a right coset of \(H\) is a subset of \(G\) of the form \(Hg = \{hg : h \in H\}\) for some \(g \in G\)). Notice that if \(G\) is abelian, there is of course no difference between left and right cosets, so we can just speak about cosets.

(a) Write down the cosets of \(\{0, 1\}\) in \(\mathbb{Z}/8\mathbb{Z}\).

(b) \((\text{NH})\) Write down the left cosets of \(\langle r \rangle\) in \(D_{2n}\), and the left cosets of \(\langle s \rangle\) in \(D_{2n}\).

(c) Let \(G\) be a group and \(H \subset G\) a subgroup. Show that any two left cosets \(gH, kH\) are either equal, or disjoint.

(d) \((\text{NH})\) Let \(Ax = b\) be a system of linear equations (written in matrix form: \(A \in M_{m \times n}(F), x \in F^n, b \in F^m\)). Define \(\text{Sol}(b) = \{x \in F^n \mid Ax = b\}\). Notice that \(\text{Sol}(0)\) is a subgroup of \(F^n\) (where the group law is addition). Show that if \(\text{Sol}(b)\) is non-empty, then it is a coset of \(\text{Sol}(0)\) (“the solutions to a nonhomogenous system of linear equations are either non-existant, or given by a special solution, plus a general solution to the corresponding homogenous system of equations”).
4. Let $G$ be a group, $X$ a set, and suppose that we are given an action of $G$ on $X$. We defined, for $x \in X$, $\text{Stab}_G(x) := \{g \in G \mid gx = x\}$. Similarly define, for $x, y \in X$, $\text{Tran}_G(x, y) := \{g \in G \mid gx = y\}$. It is a subset of $G$ (usually not a subgroup!).

(a) (NH) Notice that $\text{Stab}_G(x) = \text{Tran}_G(x, x)$.

(b) (NH) Notice that $\text{Tran}_G(x, y) \neq \emptyset$ if and only if $x, y$ sit in the same orbit of the action.

(c) Suppose that $x, y$ sit in the same orbit of the action. Show that $\text{Tran}_G(x, y)$ is a left coset of $\text{Stab}_G(x)$, and also a right coset of $\text{Stab}_G(y)$.

(d) Suppose that $x, y$ sit in the same orbit of the action. Show that the subgroups $\text{Stab}_G(x), \text{Stab}_G(y) \subset G$ are conjugate (two subgroups $H, K \subset G$ are called conjugate if there exists $g \in G$ such that $H = gKg^{-1} = \{gkg^{-1} : k \in K\}$).

(e) (NH) Let $A \in M_{m \times n}(F)$ be a matrix. Define an action of $F^n$ on $F^m$ by: $v \otimes w := Av + w$. Interpret the space of solutions to $Ax = b$ as $\text{Tran}(0, b)$. Realize that in this case, item (c), saying that $\text{Tran}(0, b)$ is either empty, or a coset of $\text{Tran}(0, 0)$, is again the basic principle of linear algebra mentioned in the previous exercise.

5. (a) (NH) Understand that $\{1, -1\} \subset Z$ is a group w.r.t. multiplication of integers. Understand that it is isomorphic to $Z/2Z$.

(b) (NH) Let $G$ be a group. Consider the action of the group $\{1, -1\}$ on the set $G$, given by $\epsilon \cdot g := g^\epsilon$. Understands the sizes of orbits, and on what they depend.

(c) Let $G$ be a finite group of even order. Show that $G$ has an element of order 2 (Hint:

(d) (NH) Later we will see, that if the order of a finite group is divisible by a prime $p$, then that group has an element of order $p$.

6. (NH)
(a) Show that $\mathbb{Z}$ has exactly the following subgroups:

- For every positive integer $n$, $\langle n \rangle$, also known as $n\mathbb{Z}$, consisting of all integers divisible by $n$.
- $0 := \{0\}$.

(b) Let $G$ be a group and $g \in G$ an element. Consider the homomorphism $a_g : \mathbb{Z} \to G$ given by $a_g(n) = g^n$. Show that if $o(g) = \infty$ then $\text{Ker}(a_g) = 0$, while if $o(g) \neq \infty$, then $\text{Ker}(a_g) = \langle o(g) \rangle$.

(c) Think about the analogy of the above with the following: When we study a linear map $T : V \to V$, an important part is studying the kernel of the map $a_T : F[X] \to \text{End}(V)$, given by $a_T(f) = f(T)$. The kernel is an ideal, generated by what we call the minimal polynomial of $T$. Thus, the minimal polynomial from linear algebra is analogous to the order from group theory. One can think of these invariants as describing as much as possible of $g/T$, when all we know is that element’s algebraic interactions with itself. In other words, as we saw, $o(g)$ describes $\langle g \rangle$ up to isomorphism (and, similarly, the minimal polynomial of $T$ describes the subalgebra of $\text{End}(V)$ generated by $T$ up to isomorphism).

7. (NH) Recall that $\mathbb{Z}/n\mathbb{Z}$ carries not only addition, about which we talked, but also multiplication (it is a so-called ”ring”). Under multiplication, it is a monoid, and taking the subset of invertible elements yields a group, denoted $(\mathbb{Z}/n\mathbb{Z})^\times$ - see Section 0.3 of the textbook (or ”Multiplicative group of integers modulo n” in Wikipedia, for example).

(a) Write down all the elements of $(\mathbb{Z}/18\mathbb{Z})^\times$.

(b) Let $G$ be a group of order $n$. Show that there is an action of $(\mathbb{Z}/n\mathbb{Z})^\times$ on $G$, given by $a \otimes g := g^a$.

8. (NH) Let $p$ be a prime number. Let $G$ be an abelian group, in which every element $g$ satisfies $g^p = 1$ (equivalently, every non-identity element has order $p$). Show that $G$ has a unique structure of a vector space over the field $\mathbb{Z}/p\mathbb{Z}$, such that the underlying abelian group law of the vector space coincides with that already on $G$.

9. (NH) Give an example of an infinite group, in which every element has finite order.
10. (NH) Fix a positive integer $n$. Give an example of a finite group, which can not be generated by $n$ elements.

11. (NH) Let $G$ be a finite group such that $|Aut(G)| = 1$. Show that $G \cong \mathbb{Z}/2\mathbb{Z}$ (Hint: 

12. (NH) Is there a non-trivial group $G$ which satisfies $G \cong G \times G$ (Remark: a non-trivial group means a group with more than one element)?