Problem Set 5, Ma 001a-sec1, Fall 2016-17
Due 4 pm Monday November 14th

Remark: This homework set has 150 points.

1. (20 points) Show that
\[
\cos \left( (j + \frac{1}{2})t \right) - \cos \left( (j - \frac{1}{2})t \right) = -2 \sin \left( \frac{1}{2}t \right) \sin (jt)
\]
and deduce that if \( t/(2\pi) \) is not an integer, then
\[
\sum_{j=1}^{n} \sin(jt) = \frac{\cos(\frac{t}{2}) - \cos \left( (n + \frac{1}{2})t \right)}{2 \sin(\frac{t}{2})}.
\]

Use this to evaluate
\[
\int_{0}^{\pi/2} \sin x \, dx.
\]

2. (15 points) Use a similar method to that of the previous problem to evaluate
\[
\int_{\pi}^{3\pi/4} \cos x \, dx.
\]

3. (15 points) Let \( f(x) = 1/x^2 \), and for any partition \( 1 = x_0 < x_1 < x_2 < \cdots < x_n = 2 \) of the interval \([1, 2]\), consider the points \( c_i = \sqrt{x_{i-1}x_i} \). Show that each \( c_i \) is in \([x_{i-1}, x_i]\) and calculate \( \sum_{i=1}^{n} f(c_i) \Delta x_i \), where \( \Delta x_i = x_i - x_{i-1} \). What can you conclude about \( \int_{1}^{2} (1/x^2) \, dx \)?

4. (10 points) Find two numbers \( x_1 \) and \( x_2 \) in the interval \([0, 1]\) with \( x_1 < x_2 \) such that for any cubic polynomial \( f(x) = ax^3 + bx^2 + cx + d \) we have
\[
\int_{0}^{1} f(x) \, dx = \frac{f(x_1) + f(x_2)}{2}.
\]

5. (15 points) Prove that if \( f \) and \( g \) are integrable functions over an interval \([a, b]\), then \( f + g \) is also integrable over the interval and
\[
\int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx.
\]

6. (15 points) Assuming that \( f'(x) = 1/x \) and \( f(2) = 9 \), find
\[
\lim_{x \to 2} \frac{f(x^2 + 5) - f(9)}{x - 2}.
\]
7. (10 points) Suppose \( f(0) = 0 \) and \( |f(x)| > \sqrt{|x|} \) for all \( x \neq 0 \). Show that \( f'(0) \) does not exist.

8. (10 points) If a function \( f \) is differentiable at a point \( x \), show that
\[
\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x).
\]

9. (20 points) Show that there is a line through the point \((a, 0)\) on the plane that is tangent to the curve \( y = x^3 \) at \( x = 3a/2 \). If \( a \neq 0 \), is there any other line through \((a, 0)\) that is tangent to the curve? If \((x_0, y_0)\) is an arbitrary point on the plane, what is the maximum number of lines through \((x_0, y_0)\) that can be tangent to \( y = x^3 \), and what is the minimum number?

10. (20 points) Compute \( \frac{dy}{dx} \) in the following cases:

   (1) \[ y = (x + \sin^5 x)^8 \]

   (2) \[ y = \frac{\sin(x^3) \sin^3 x}{1 + \sin x} \]

   (3) \[ y = \sin \left( \frac{x^3}{\sin (\frac{x^3}{\sin x})} \right) \]

   (4) \[ 3x^3 + 4x^2 y - xy^2 + 2y^5 = 4 \]