Problem Set 2, Ma 001a-sec1, Fall 2016-17
Due 4 pm Monday October 10th

1. (20 points) Compute the following limit directly and interpret the final answer as a specific area:
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( a + \frac{i(b-a)}{n} \right) \frac{3}{n} b - a.
\]

2. (10 points) Consider the functions \( f(x) = \sin(1/x) \) defined for \( x \neq 0 \). Show that it is not possible to define \( f(0) \) in a way that \( f \) becomes continuous at 0.

3. (20 points) Find the following limits and use the \( \varepsilon - \delta \) definition to show that your answer is correct
   
   (1) \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) \)
   
   (2) \( \lim_{x \to a} \frac{x^n - a^n}{x-a} \), where \( n \) is a positive integer.
   
   (3) \( \lim_{x \to 0} \frac{1 - \sqrt{1-x^2}}{x^2} \)
   
   (4) \( \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x-a} \), where \( a \) is a positive real number.

4. (10 points) If \( \lim_{x \to 0} \frac{f(x)}{x} = -5 \), find \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} \frac{f(x)}{x} \).

5. (20 points) Prove the following statements:
   
   (1) Assume that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function such that
       \( f(x+y) = f(x) + f(y) \), for any real numbers \( x, y \).
       Show that \( f(x) = cx \) for any real number \( x \), where \( c \) is a fixed constant.
       (Hint: prove it for rational numbers first.)
   
   (2) Assume that \( g : \mathbb{R} \to \mathbb{R} \) is a continuous function such that
       \( g \left( \frac{x+y}{2} \right) = \frac{g(x) + g(y)}{2} \), for any real numbers \( x, y \).
       Show that \( g(x) = cx + b \) for any real number \( x \), where \( c \) and \( b \) are fixed constants.

6. (20 points) Assuming that \( \lim_{x \to p} g(x) = B \neq 0 \), show that
   \[
   \lim_{x \to p} \frac{1}{g(x)} = \frac{1}{B}.
   \]