1. (25 points) Prove the binomial theorem by induction: for any positive integer $n$ and real numbers $a$ and $b$ we have

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$ 

Here the binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where we are using the following recursive definition for the factorial function $n!$:

$0! = 1, \quad 1! = 1, \quad 2! = 2 \times 1! = 2, \quad 3! = 3 \times 2! = 6, \quad 4! = 4 \times 3! = 12, \quad n! = n \times (n-1)!$

Then, show that

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$ 

2. (25 points) Use induction to show that for any pair of positive integers $n, k$, we have

$$1^k + 2^k + \cdots + (n-1)^k < \frac{n^{k+1}}{k+1} < 1^k + 2^k + \cdots + n^k.$$ 

3. (25 points) Show by induction that for any positive integer $n$ we have

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$ 

4. (25 points) Consider the Fibonacci numbers $F_n$:

$$F_1 = 1, \quad F_2 = 2, \quad F_3 = F_1 + F_2 = 3, \quad F_4 = F_3 + F_2 = 5, \quad F_5 = F_4 + F_3 = 8, \quad \ldots$$

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 3.$$

Show by induction that for any positive integer $n$, we have

$$F_n < \left(\frac{1 + \sqrt{5}}{2}\right)^n.$$