(1) Two sets $x, y$ are called almost disjoint if their intersection is finite. Show that there is a collection $\mathcal{A}$ of $2^{\aleph_0}$ subsets of $\omega$ which are almost disjoint.

**Solution.** For each $f : \omega \to 2$, let $I_f = \{f|n : n \in \omega\} \subseteq 2^{<\omega}$ where $f|n = (f(0), \ldots, f(n-1))$. If $f \neq g$ then $I_f \cap I_g$ is finite, so that $A = \{I_f : f \in 2^\omega\}$ is a collection of $2^{\aleph_0}$ subsets of $2^{<\omega}$ which are almost disjoint. We are then done by taking $A = \{\varphi(I_f) : I_f \in A\}$, where $\varphi : 2^{<\omega} \to \omega$ is any bijection.

(2) Prove that we can decompose $\mathbb{R}$ into $2^{\aleph_0}$ many sets each of which has cardinality $2^{\aleph_0}$.

**Solution.** Fix a bijection $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (Note: $\mathbb{R} \times \mathbb{R} \approx 2^\omega \times 2^\omega \approx 2^\omega \approx \mathcal{P}(\mathbb{R})$). For each $r \in \mathbb{R}$, let $I_r = \{\overline{r}\} \times \mathbb{R}$. Then $I_r \subseteq \mathbb{R} \times \mathbb{R}$ and the collection $\{I_r : r \in \mathbb{R}\}$ decomposes $\mathbb{R} \times \mathbb{R}$ into $2^{\aleph_0}$ many sets each of which has cardinality $2^{\aleph_0}$. Thus $\{\psi(I_r) : r \in \mathbb{R}\}$ is such a decomposition for $\mathbb{R}$.

(3) Show that $\mathbb{R}^\mathbb{R} \approx \mathcal{P}(\mathbb{R})$.

**Solution.** For any set $X$ we have that $2^X \approx \mathcal{P}(X)$. We will also use $\mathbb{R} \approx 2^\omega$. Write $X \subseteq Y$ to mean that there exists a 1-1 function $X \to Y$. We have that $\omega \times 2^\omega \approx 2^\omega$ since

$$2^\omega \subseteq \omega \times 2^\omega \subseteq 2^\omega \times 2^\omega \approx 2^\omega.$$  

Thus

$$\mathbb{R}^\mathbb{R} \approx (2^\omega)^2^\omega \approx 2^{\omega \times 2^\omega} \approx 2^{2^\omega} \approx 2^\mathbb{R} \approx \mathcal{P}(\mathbb{R}).$$

(4) A Cantor scheme on a set $X$ is a family $(P_s)_{s \in 2^{<\omega}}$ of subsets of $X$, indexed by the set of all finite binary sequences $2^{<\omega} = \bigcup_n 2^n$, such that

- $P_{s-0} \cap P_{s-1} = \emptyset$ for any $s \in 2^{<\omega}$,
- $P_{s-1} \subseteq P_s$, for any $s \in 2^{\leq n}, i \in \{0, 1\}$.

Such a Cantor scheme determines the following subset of $X$:

$$A_s P_s = \bigcup_{x \in 2^n} \bigcap_{n \in \mathbb{N}} P_{x|n}.$$  

Show that

$$A_s P_s = \bigcap_{n \in \mathbb{N}} \bigcup P_s.$$  

Moreover show that if each $P_s$ is a perfect nonempty subset of $\mathbb{R}$ and the diameter of $P_{x|n}$ goes to 0 as $n \to \infty$, for each $x \in 2^N$, then $A_s P_s$ is also perfect nonempty.

**Solution.** Let $A = \bigcup_{x \in 2^n} \bigcap_{n \in \mathbb{N}} P_{x|n}$ and let $B = \bigcap_{n \in \mathbb{N}} \bigcup_{s \in 2^n} P_s$. We must show $A = B$. The easy containment is $A \subseteq B$: If $a \in A$, then there is an $x \in 2^N$ such that for every $n \in \mathbb{N}$, $a \in P_{x|n}$, so that in particular for every $n \in \mathbb{N}$ we have $a \in P_s$ for some $s \in 2^n$ (namely, $s = x|n$), hence $a \in B$. For the other containment, suppose $b \in B$ so that for each $n$ there exists an $s \in 2^n$ with $b \in P_s$. Form the tree

$$T = \{s \in 2^{<\omega} : b \in P_s\}.$$  


Then $T$ is an infinite, finitely branching tree so by König’s lemma has an infinite branch $x$ with $b \in P_{x|n}$ for all $n$. Hence $b \in A$, as witnessed by $x$.

Now suppose each $P_s$ is a perfect nonempty subset of $\mathbb{R}$ and that for each $x \in 2^\mathbb{N}$ the diameter of $P_{x|n}$ goes to 0 as $n \to \infty$. By compactness, the intersection $\bigcap_{n \in \mathbb{N}} P_{x|n}$ is nonempty, and by the vanishing diameter condition this intersection is in fact a singleton, say $\bigcap_{n \in \mathbb{N}} P_{x|n} = \{f(x)\}$. This defines a function $f : 2^\mathbb{N} \to \mathbb{R}$. This is a 1-1 function (by the first defining property of a Cantor scheme), and is continuous. We have that $A_s P_s = f(2^\mathbb{N})$, and $f(2^\mathbb{N})$ is a compact subset of $\mathbb{R}$ that is homeomorphic to $2^\mathbb{N}$ (since $2^\mathbb{N}$ is compact Hausdorff), hence is perfect, nonempty.

**Alternative proof that does not assume as much knowledge of topology:** That $A_s P_s$ is nonempty is observed above. Since $A_s P_s = \bigcap_{n \in \mathbb{N}} \bigcup_{s \in 2^n} P_s$ and each $\bigcup_{s \in 2^n} P_s$ is closed, so is $A_s P_s$. To show that $A_s P_s$ has no isolated points, let $a \in A_s P_s$, so that $a \in \bigcap_{n \in \mathbb{N}} P_{x|n}$ for some $x \in 2^\mathbb{N}$, and let $I$ be an open interval around $a$. Then for large enough $n$, $P_{x|n} \subseteq I$. If $x(n) = i$, and $1 - x(n) = j$ then $a \in P_{x|n-i}$ and there is $b \in A_s P_s$ with $b \in P_{x|n-j} \subseteq I$, so that $a \neq b$.

(5) Show that if $f : [0, 1] \to \mathbb{R}$ is continuous, either $f$ is constant in some subinterval $[a, b]$ of $[0, 1]$ or else there is a nonempty perfect subset $P \subseteq [0, 1]$ such that $f$ is 1 − 1 on $P$.

**Solution.** Suppose $f$ is not constant on any subintervals of $[0, 1]$. We will construct a Cantor scheme $(P_s)$, where each $P_s$ is a closed subinterval of $[0, 1]$ of length no greater than $2^{-|s|}$, where $|s| = n$ if $s \in 2^n$. Let $P_0 = [0, 1]$. Let $s \in 2^{<\mathbb{N}}$ and assume that $P_s$ has been constructed as described, i.e. $P_s = [a, b]$ ($a < b$) is a closed subinterval of $[0, 1]$. Since $f$ is not constant on any subintervals, there are $a < x_0 < x_1 < b$ with $f(x_0) \neq f(x_1)$, so let $P_{s-0}$ and $P_{s-1}$ be closed intervals containing $x_0$ and $x_1$, respectively, of length $< 2^{-|s|+1}$ and such that $f(P_{s-0}) \cap f(P_{s-1}) = \emptyset$ (these can be found as $f$ is continuous). This last condition ensures that $f$ is injective on $A_s P_s$, and by problem (4*), $A_s P_s$ is a nonempty perfect subset of $[0, 1]$.