1. Let $G = (V, E)$ be a planar graph with $|V| \leq 11$. Prove that the vertices of $G$ can be colored using four colors. You are not allowed to rely on the four color theorem.

2. Let $\mathcal{P}$ be a set of $n$ points in the plane, such that the distance between any two points of $\mathcal{P}$ is at least 1. Prove that there must exist a set of $\lceil n/4 \rceil$ points $\mathcal{P}' \subset \mathcal{P}$, such that every two points of $\mathcal{P}'$ are at a distance greater than one from each other (hint: Color the vertices of a graph).

3. Let $r_k = R(3, \ldots, 3; 2)$. Prove that for every $k \geq 3$, we have $r_k \leq k(r_{k-1} - 1) + 2$.

4. In class we proved that in any set of $R(m, 5; 4)$ points in the plane (with no three on a line) we can find the vertices of a convex $m$-gon. Prove that this is also the case for any set of $R(m, m; 3)$ points in the plane with no three on a line.

One way to solve the problem is to rely on the following. Let $p_1, p_2, p_3$ be three points in the plane, such that $p_1$ has the smallest $x$-coordinate, $p_2$ has the second smallest $x$-coordinate, and $p_3$ has the largest $x$-coordinate. We travel in a straight line from $p_1$ to $p_2$, and then in a straight line from $p_2$ to $p_3$. Notice that in this trip, when visiting $p_2$ we either perform a right turn or a left turn. An example is illustrated in the following figure.

Consider the points $p_1, p_2, \ldots, p_k$ in the plane, also ordered in increasing $x$-coordinate values. What happens when for every $1 \leq i \leq k - 2$ the triple $(x_i, x_{i+1}, x_{i+2})$ forms a right turn?

5. Prove that every graph $G = (V, E)$ can be turned into a tripartite graph by removing at most $|E|/3$ of the edges of $E$. Use a probabilistic proof (a non-probabilistic proof may get zero points).

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1Similarly to a bipartite graph, a tripartite graph has three vertex sets $V_1, V_2, V_3$ and no edges between vertices of the same set.