Ma/CS 6b: Problem Set 3*

Due noon, Tuesday, February 2nd

1. (a) Consider a plane graph $G = (V, E)$ (that is, a specific drawing of $G$ with no crossings). Let $T \subset E$ be the set of edges of a specific spanning tree of $G$. Prove that the dual edges of $E \setminus T$ form a spanning tree in $G^*$.

(b) Let $G = (V, E)$ be a connected plane graph that is identical to its dual $G^* = (V^*, E^*)$. That is, there exists a bijection $f : V \to V'$ such that $(v, u) \in E$ if and only if $(f(v), f(u)) \in E'$. Express $|E|$ as a function of $|V|$.

2. Let $G$ be a connected planar graph with $n$ vertices and girth four. Find a tight bound for the maximum number of edges in $G$ (as a function of $n$). Prove your answer.

3. Let $P$ be a set of $n \geq 3$ points in the plane, such that the distance between any two points of $P$ is at least 1. Prove that at most $3n - 6$ pairs of points of $P$ are at a distance of exactly one from each other.

4. Let $G = (V, E)$ be a connected graph with no $K_4$ and $K_{2,3}$ topological minors. Prove that there exists a drawing of $G$ with no edge crossings and with every vertex of $V$ on the boundary of a specific face $f$ (hint: Use Kuratowski’s theorem).

5. Consider a graph $G = (V, E)$ with $|V| \geq 5$ and an edge $e \in E$ such that after removing $e$ from $G$ it becomes a triangulation (so $G$ is not a planar graph). Prove that $G$ contains $K_5$ as a topological minor. You may rely on the claim “every triangulation with at least five vertices is 3-connected” without proving it (hint: Notice that the neighbors of a vertex in a triangulation form a cycle).

*The awesome students who helped correcting this assignment: Eugene Bulkin.

†The girth of a graph was defined in the first problem set.