Ramsey Numbers (Special Case)

- $p_1, \ldots, p_k$ – positive integers.
- The **Ramsey number** $R(p_1, \ldots, p_k; 2)$ is the minimum integer $n$ such that for every coloring of the edges of $K_n$ using $k$ colors, at least one of the following subgraphs exists:
  - A $K_{p_1}$ colored only with color 1.
  - A $K_{p_2}$ colored only with color 2.
  - ...
  - A $K_{p_k}$ colored only with color $k$. 
An Example

- The expression $n = R(3,3,3,3; 2)$ means that every coloring of the edges of $K_n$ using four colors contains a monochromatic triangle.

Another Example

- We already proved that $R(3,3; 2) = 6$.
- In the 6-people problem we proved that any coloring $K_6$ using two colors contains a monochromatic triangle.
  - This is not the case for $K_5$. 
Recall: Asymptotic Bounds

- By using
  \[ R(p_1, p_2) \leq R(p_1 - 1, p_2) + R(p_1, p_2 - 1) \]
  one can show that \( R(p, p) \leq c \frac{4^p}{\sqrt{p}} \) (for some constant \( c \)).

- We also proved \( R(p, p) > 2^{p/2} \).

Homogenous Subsets

- \( S \) – a set of “elements” (e.g., vertices).
  - For an integer \( 1 \leq r \leq |S| \), we let \( \binom{S}{r} \) the set of all subsets of \( r \) elements of \( S \).
  - We give every subset of \( \binom{S}{r} \) a color.
  - We say that \( S' \subset S \) is \textit{i-homogeneous} if every subset of \( \binom{S'}{r} \) is of color \( i \).
Example: Subsets of Size 2

- When the subsets are of size two, this is equivalent to **coloring edges of a graph**.

![Graphs showing subsets of size 2](image)

General Ramsey Numbers

- $r, p_1, \ldots, p_k$ – positive integers.
- The **Ramsey number** $R(p_1, \ldots, p_k; r)$ is the minimum integer $n$ such that every coloring of $\binom{\{1, 2, \ldots, n\}}{r}$ using $k$ colors yields an **$i$-homogeneous set of size $p_i$**, for some $1 \leq i \leq k$. 
A Bit of Intuition

- If $r = 3$, we color triples of vertices.
  - Can be thought of as coloring the triangular faces of $K_n$.
  - We are looking for a large subset $S$ of the vertices, such that each triangle that is spanned by three vertices of $S$ has the same color.

![Monochromatic $K_4$](image)

The Case of Larger $r$

- We already know that $R(p_1, p_2; 2)$ is finite for any positive $p_1$ and $p_2$.
  - We now generalize this to the case of larger $r$.
- **Theorem.** For any positive integers $r, p_1, p_2$, the Ramsey number $R(p_1, p_2; r)$ is finite.
What Do We Need to Prove?

- For any positive integers \( r, p_1, p_2, \) we need to prove that there exists \( n \) such that every coloring of the \( r \)-tuples of a set of size \( n \) in red and blue contains:
  - Either a subset of \( p_1 \) elements such that every \( r \)-tuple in it is red,
  - or a subset of \( p_2 \) elements such that every \( r \)-tuple in it is blue.

Proof

- We use a \textit{double induction}.
  - We prove the theorem by induction on \( r \).
  - We prove the induction step using an induction on \( p_1 + p_2 \).

- Induction basis (on \( r \)).
  - \textbf{When} \( r = 1 \), we can simply take \( p_1 + p_2 - 1 \) elements.
  - We already proved the case of \( r = 2 \).
Induction Step

- We prove the induction step for a given value of $r$ by induction on $p_1 + p_2$.
  - **Induction basis.** If $p_1 < r$ or $p_2 < r$ then the claim vacuously holds for sets of $p_i$ elements and no $r$-tuples.
  - **Induction step.** We set $q_1 = R(p_1 - 1, p_2; r)$, $q_2 = R(p_1, p_2 - 1; r)$, and $n = 1 + R(q_1, q_2; r - 1)$.
  - By the hypotheses, $q_1, q_2$, and $n$ are finite.

Proof (cont.)

$$q_1 = R(p_1 - 1, p_2; r), \quad q_2 = R(p_1, p_2 - 1), \quad n = 1 + R(q_1, q_2; r - 1).$$

- Let $S$ be a set of $n$ elements, with every $r$-tuple colored either red or blue.
  - Pick an element $x \in S$ and let $S' = S \setminus \{x\}$.
  - We color an $(r - 1)$-tuple $T \subset S'$ using the same color as the $r$-tuple $T \cup \{x\}$.
  - Since $|S'| = n - 1 = R(q_1, q_2, r - 1)$, there is either a red subset of $q_1$ elements (with respect to the colors of the $(r - 1)$-tuples), or a blue subset of $q_2$ elements.
Completing the Proof

\[ q_1 = R(p_1 - 1, p_2; r), \quad q_2 = R(p_1, p_2 - 1), \]
\[ n = 1 + R(q_1, q_2, r - 1). \]

- \( S \) – a set of \( n \) elements. \( S' = S \setminus \{x\} \).
- WLOG, we assume that there is a red subset \( S_r \) of \( q_1 \) elements (with respect to \((r - 1)\)-tuples).
- We consider the \( r \)-tuples of \( S_r \). Since \(|S_r| = q_1 = R(p_1 - 1, p_2; r)\), either there is a blue subset of size \( p_2 \), or a red subset of size \( p_1 - 1 \).
- In the latter case, by adding \( x \) to the subset, we obtain a red subset of size \( p_1 \).

Ramsey’s Theorem

- A straightforward extension of the proof yields a more general result.
- **Theorem.** For any positive integers \( r, p_1, \ldots, p_k \), the Ramsey number \( R(p_1, \ldots, p_k; r) \) is finite.
  - One way to think of the theorem: in every sufficiently large arbitrary object (i.e., an arbitrary coloring) there must be some order (i.e., a monochromatic subset).
Some History

- In the early 1930’s in Budapest, a few students used to regularly meet on Sundays in a specific city park bench.
- Among the participants were Paul Erdős, George Szekeres, and Esther Klein.
- Klein told the group that for any set of five points with no three on a line, four of the points are the vertices of a convex quadrilateral.

- The **convex hull** of a point set $S$ is the smallest convex polygon that contains $S$.
- Given a set of five points:
  - If the convex hall of the point set has at least four points in it, then we are done.
  - If the convex hull consists of three points, we consider the line $\ell$ that passes through the two interior points.
  - We take the two interior points and the two points that are on the same side of $\ell$. 
The Story Continues

- The Budapest students started to think about whether there exists an $n$ such that every set of $n$ points with no three on a line contains the vertices of a convex pentagon.
  - More generally, does a similar condition hold for every convex $k$-gon?

The Happy Ending Problem

- Even though Erdős was part of the group, the first to prove the claim was Szekeres.
- A couple of years later, Esther Klein, who suggested the problem married George Szekeres who solved it.
  - Since then, this problem is called "the happy ending problem".
  - They lived together up to their 90’s.
The Theorem

- **Theorem.** For every integer \( m \geq 3 \), there exists an integer \( N(m) \) such that any set of \( N(m) \) points with no three on a line contains a subset of \( m \) points that are the vertices of a convex \( m \)-gon.

First Claims

- **Straightforward claim.** Any four vertices of a convex \( n \)-gon span a convex quadrilateral.
- **Less straightforward claim.** If every four vertices of an \( n \)-gon \( P \) form a convex quadrilateral, then \( P \) is convex.
Proving the Claim

- **Claim.** If every four vertices of an \( n \)-gon \( P \) form a convex quadrilateral, then \( P \) is convex.

- **Proof.** Assume *for contradiction* that \( P \) is not convex.
  - There is a vertex \( v \) that is not in the convex hull of the vertices of \( P \).
  - Triangulate the convex hull of \( P \).
  - \( v \) together with the three vertices of the triangle containing \( v \) do not span a convex quadrilateral!

Proving the Theorem

- Set \( N(m) = R(m, 5; 4) \).
  - Given a set of \( N(m) \) points with no three on a line, we color every 4-tuple of points.
  - A 4-tuple is colored *red* if it spans a convex quadrilateral, and otherwise *blue*.
  - By Ramsey’s theorem, either there is a *red subset of size* \( m \) and or *blue one of size* 5.
  - But we proved that for any 5 points there must be a convex (=red) quadruple!
  - Thus, there is a *red* subset of size \( m \), and by the previous claim it spans a convex \( m \)-gon.
The Erdős-Szekeres Conjecture

- In the 1930’s, Erdős and Szekeres proved \( 2^{m-2} + 1 \leq N(m) \leq \left( \frac{2m-4}{m-2} \right) + 1 \).
- In 2005, Tóth and Valtr improved this to \( N(m) \leq \left( \frac{2m-5}{m-2} \right) + 1 \).
- Conjecture (Erdős and Szekeres).
  \[ N(m) = 2^{m-2} + 1. \]
  - Using computers, this was verified for \( m \leq 6 \).

Frank P. Ramsey

- Died in 1930 at the age of 26. By then, he:
  - Wrote mathematical papers, including getting a whole subfield named after him.
  - Wrote several philosophical works. Wittgenstein mentions him in the introduction to his *Philosophical Investigations* as an influence.
  - Wrote several economics papers, as a student to John Maynard Keynes.
  - Had a wife, kids, etc.
Graph Ramsey Theory

- So far we looked for monochromatic copies of some $K_m$ in colorings of $K_n$.
  - Searching monochromatic copies of other types of graphs has also been studied.
  - Given two graphs $G_1, G_2$, we denote by $R(G_1, G_2)$ the minimum number $n$ such that every coloring of $K_n$ contains either a blue copy of $G_1$ or a red copy of $G_2$.

Example.
- $P_m$ – a graph that is a path of length $m$.
- Then $R(P_2, P_2) = 3$.

The Case of a Tree

- **Theorem.** Let $T$ be a tree with $m$ vertices. Then $R(T, K_n) = (m - 1)(n - 1) + 1$.
- **Proof.** We begin with a lower bound.
  - Consider $n - 1$ copies of $K_{m-1}$ colored completely in blue. Edges between vertices of different copies are colored red.
  - This is a set of $(m - 1)(n - 1)$ vertices containing no red $K_n$ and no blue $T$.
  - Thus, $R(T, K_n) > (m - 1)(n - 1)$. 
Proof: Upper Bound

• **Claim.** Let $T$ be a tree with $m$ vertices. Then $R(T, K_n) = (m - 1)(n - 1) + 1$.

• **Proof.** We prove the upper bound by induction on $m + n$.
  - **Induction basis:** if $m = 1$ or $n = 1$, the claim obviously holds.
  - **Induction step:** Set $N = (m - 1)(n - 1) + 1$. Consider a coloring of $K_N$ and a vertex $v$.
    - If $v$ is incident to more than $(m - 1)(n - 2)$ red edges, by the hypothesis, either the neighbors span a blue $T$ or $v$ and its neighbors span a red $K_n$.

Proof: Upper Bound (cont.)

• We set $N = (m - 1)(n - 1) + 1$.

• If any vertex is incident to more than $(m - 1)(n - 2)$ red edges, we are done.

• Assume that every vertex is incident to at most $(m - 1)(n - 2)$ red edges.

• That is, every vertex is adjacent to at least $m - 1$ blue edges.

• In the first assignment, we proved that any graph with minimum degree at least $m - 1$ contains every tree with $m$ vertices.
Copies of Triangles

- We denote as $mK_3$ a set of $m$ disjoint copies of $K_3$.

**Theorem.** $R(mK_3, mK_3) = 5m$, for every $m \geq 2$.

**Proof.** We begin with the lower bound.

- Consider a red $K_{3m-1}$ and another red $K_{1,2m-1}$. Every other edge in the graph is blue.
- There are $5m - 1$ vertices, with $m - 1$ disjoint red triangles and $m - 1$ disjoint blue triangles.
- Thus, $R(mK_3, mK_3) > 5m - 1$.

Illustration
Proof: Upper Bound

- **Theorem.** $R(mK_3, mK_3) = 5m$, for $m \geq 3$.

- **Proof.** We prove an upper bound by induction on $m$.

  - **Induction basis.** The case of $m = 2$ is **not trivial**, but we will not do it.

  - **Induction step.** Consider a coloring of $K_{5m}$.

    - We repeatedly look for a monochromatic triangle and remove its vertices from the graph.

    - Since $R(3,3) = 6$, this process continues as long as at least six vertices remain.

    - Since $5m - 3m \geq 6$ for $m \geq 3$, we have at least $m$ disjoint monochromatic triangles.

Proof: Upper Bound (cont.)

- We consider a coloring of $K_{5m}$ and find $m$ monochromatic triangles in it.

- If all triangles are of the same color, we are done. Thus, assume that we have a red triangle $\Delta abc$ and a blue triangle $\Delta def$.

- WLOG, assume that at least 5 of the 9 edges between the two triangles are red.

- WLOG, assume that two of these 5 edges meet in $d$. We thus have a red triangle and a blue triangle with a common vertex $d$. 

![Diagram](image)
Proof: Upper Bound (cont.)

- We consider a coloring of $K_{5m}$.
- It remains to consider the case of a red triangle and a blue triangle have a common vertex.
- By the induction hypothesis, the remaining $5m - 5$ vertices contain $m - 1$ disjoint triangles of the same color.
- By adding one of the two triangles, we obtain $m$ triangles of the same color.

The End