Ma/CS 6b
Class 12: Ramsey Theory

By Adam Sheffer

The Pigeonhole Principle

• **The pigeonhole principle.** If \( n \) items are put into \( m \) containers, such that \( n > m \), then at least one container contains more than one item.
Counting Hairs

- **Prove.** There exist two people such that both live in London and have exactly the same number of hairs on their head at this specific moment.

Solution

- Population of London: about 8.3 million.
- Average amount of hairs on head (according to a random website):
  - Blondes: 140,000.
  - Black hair: 110,000.
  - Red hair: 90,000.
- For simplicity, assume that no person has more than 1,000,000 hairs.
  - By the pigeonhole principle, there must be at least eight people with London, all with the same number of hairs.
An Erdös Initiation Question

- **Prove.** Consider a subset $A \subset \{1, 2, 3, \ldots, 2n\}$ with $|A| = n + 1$. Then there exist $b, c \in A$ such that $b$ divides $c$.

- **Answer.** Write every $a \in A$ as $2^k m$, where $m$ is odd.
  - There are $n$ possible values for $m$.
  - By the pigeonhole principle, there are two elements of $A$ with the same $m$. One of those divides the other.

Mutual Acquaintances

- **Prove.** Among every six people, it is possible to find either three mutual acquaintances or three non-mutual acquaintances.

- **Solution.** We build a graph.
  - A vertex for every person.
  - An edge between every pair of people who know each other.
Rephrasing the Problem

- **Prove.** In a graph with six vertices, there is either a cycle of length three or three vertices with no edges between them.

- **Solution.** Consider a vertex $v$.
  - By the pigeonhole principle, there are either at least three vertices adjacent to it, or at least three vertices that are not.

Solution

- Say that $v$ is connected to three of the other vertices $u_1, u_2, u_3$.
  - If no edge exists in the subgraph induced by $u_1, u_2, u_3$, then we are done.
  - If there exists an edge between $u_i$ and $u_j$, then $\{v, u_i, u_j\}$ is a cycle of length three.
  - The case where $v$ is not connected to three vertices is treated symmetrically.
Reminder: Hypercube Graphs

- The \(d\)-dimensional hypercube graph \(Q_d\).
  - Every vertex is a point with \(d\) coordinates, each either 0 or 1.
  - Two vertices are adjacent if they have \(d - 1\) common coordinates.
  - \(2^d\) vertices.
  - \(2^{d-1}d\) edges.

Spanning Trees in Hypercubes

- **Theorem.** Let \(T\) be a spanning tree of the hypercube graph \(Q_d = (V, E)\). Then there exists an edge \(e \in E\) such that inserting \(e\) to \(T\) yields a cycle of length at least \(2d\).
Solution

- We say that two vertices $v, v' \in V$ are **opposite** if they differ in all $d$ coordinates.
  - For every $v \in V$, we consider the path in $T$ from $v$ to its opposite $v'$. We **direct** the first edge in this path so that it leaves $v$.
  - A spanning tree has $|V| - 1$ edge, and we just directed $|V|$ of those, so by the pigeonhole principle there is an edge $e \in E$ that was directed twice!

Solution (cont.)

- $(u, v) \in E$ – an edge that was **directed twice**.
  - It is the first edge in the path in $T$ from $v$ to $v'$ and the first edge in the path in $T$ from $u$ to $u'$.
  - Thus, the path from $v$ to $u'$ and the path from $u$ to $v'$ are disjoint.
  - Each of these paths is of length **at least $d - 1$** since $v, v'$ and $u, u'$ are opposite pairs.
  - Adding the edge $(v', u')$ yields a cycle of length at least $2d$ ($(v', u') \in E$ since there is an edge between the two opposite vertices).
An Alternative Formulation

- **Pigeonhole principle.** If there are \( n \) pigeons and \( n + 1 \) holes, then at least one pigeon must have at least two holes in it.

Ramsey Numbers (Special Case)

- \( p_1, \ldots, p_k \) – positive integers.
- The **Ramsey number** \( R(p_1, \ldots, p_k; 2) \) is the minimum integer \( n \) such that for every coloring of the edges of \( K_n \) using \( k \) colors, at least one of the following subgraphs exists:
  - A \( K_{p_1} \) colored only with color 1.
  - A \( K_{p_2} \) colored only with color 2.
  - ...
  - A \( K_{p_k} \) colored only with color \( k \).
An Example

- The expression \( n = R(3,3,3,3; 2) \) means that every coloring of the edges of \( K_n \) using four colors contains a monochromatic triangle.

Another Example

- We already proved that \( R(3,3; 2) = 6 \).
- In the 6-people problem we proved that any coloring \( K_6 \) using two colors contains a monochromatic triangle.
  - This is not the case for \( K_5 \).
A Special Case

• In this class we only study the case of two colors.
  ◦ For simplicity, we write $R(r, s)$ instead of $R(r, s; 2)$.

$R(2,4)$

• What is $R(2,4)$?
  ◦ Either there is a red $K_2$, which is just a red edge, or there is a blue $K_4$.
  ◦ This is the case for any coloring of $K_4$, but not for any coloring of $K_3$.
  ◦ Thus, $R(2,4) = 4$. 
Fighting Aliens

“Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens.”

Joel Spencer

The Known Bounds

• Best known bounds for $R(r, s)$:

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<th>3</th>
<th>4</th>
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<th>6</th>
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<td>4 1</td>
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<td>9</td>
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<td>5</td>
<td>14</td>
<td>25</td>
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<tr>
<td>6 1</td>
<td>6</td>
<td>18</td>
<td>36–41</td>
<td>58–87</td>
<td>102–165</td>
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<tr>
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<td>7</td>
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<td>80–143</td>
<td>113–298</td>
<td>205–540</td>
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* Taken from Wikipedia
\( \mathcal{R}(3,4) \)

- The coloring of \( K_8 \) below contains no blue \( K_3 \) and no red \( K_4 \). Therefore, \( \mathcal{R}(3,4) > 8 \).

- **Claim.** \( \mathcal{R}(3,4) = 9 \).
  - Consider any red-blue coloring of the edges of \( K_9 \) and pick any vertex \( u \).
  - Assume that \( u \) is incident to the vertices \( \{v_1, ..., v_4\} \) by red edges.
  - Since \( \mathcal{R}(2,4) = 4 \), either there is a blue \( K_4 \) or there is a red \( K_3 \) containing \( u \).

**\( \mathcal{R}(3,4) \) Analysis**

- Consider any red-blue coloring of the edges of \( K_9 \), and pick any vertex \( u \).
  - If \( u \) is incident to **four red edges**, we are done.
  - Assume that \( u \) is incident to the vertices \( \{v_1, ..., v_6\} \) by blue edges. Since \( \mathcal{R}(3,3) = 6 \), either we have a red \( K_3 \), or we have a blue \( K_4 \) containing \( u \).
  - Since \( \deg u = 8 \), it remains to consider the case where \( u \) is incident to **three red edges** and to **five blue edges**.
Completing the $R(3,4)$ Analysis

- It remains to consider the case where every vertex of $K_9$ is incident to exactly three red edges.
  - By deleting the blue edges, we remain with nine vertices of degree three.
  - This is impossible since the sum of the degrees must be even! Thus, this case cannot happen.

A General Relation

- Claim.
  \[ R(p_1, p_2) \leq R(p_1 - 1, p_2) + R(p_1, p_2 - 1). \]
- Proof. We take use colors red and blue.
  - Set \( n = R(p_1 - 1, p_2) + R(p_1, p_2 - 1) \)
  - We need to show that in any edge coloring of $K_n$ either there is a red $K_{p_1}$, or there is a blue $K_{p_2}$. 

\( R(p_1, p_2) \leq R(p_1 - 1, p_2) + R(p_1, p_2 - 1). \)

- Set \( s = R(p_1 - 1, p_2), \) \( t = R(p_1, p_2 - 1). \)
  - \( n = s + t. \) Consider a red-blue coloring of \( K_n. \)
    - Let \( v \) be a vertex in \( K_n. \)
  - Since there are \( s + t - 1 \) other vertices in \( K_n, \)
    - \( v \) is either incident to \textbf{at least} \( s \) \textbf{red edges} or to \textbf{at least} \( t \) \textbf{blue edges}.
  
- Assume that \( v \) is incident to at least \( s \) \textbf{red edges}. Thus, the “\textbf{red neighbors}” of \( v \) either contain a \textbf{red} \( K_{p_1-1} \) or a \textbf{blue} \( K_{p_2}. \)
  - In the latter case we are done. In the former, adding \( v \) yields a \textbf{red} \( K_{p_1}. \)

**Illustration**

- If \( v \) is incident to at least \( s \) \textbf{red edges}, we consider the other endpoint of these edges.
- If they contain a \textbf{red} \( K_{p_1-1} \), we add \( v \) to it.
Asymptotic Bounds

- By using
  \[ R(p_1, p_2) \leq R(p_1 - 1, p_2) + R(p_1, p_2 - 1) \]
  one can show that \( R(p, p) \leq c \frac{4p}{\sqrt{p}} \) (for some constant \( c \)).

- Theorem. \( R(p, p) > 2^{p/2} \).

Proof

- Consider all \( 2^{\binom{n}{2}} \) red-blue colorings of \( K_n \).
- A subset of \( p \) vertices of \( K_n \) forms a red or blue \( K_p \) in \( 2^{\binom{n}{2} - \binom{p}{2} + 1} \) of these colorings.
- There are at most \( \binom{n}{p} 2^{\binom{n}{2} - \binom{p}{2} + 1} \) colorings that contain a monochromatic \( K_p \).
- Thus, if \( \binom{n}{p} 2^{-(p^2)} < 1 \), there must be colorings with no monochromatic \( K_p \).
- This happens approximately when \( n < 2^{p/2} \), so \( R(p, p) > 2^{p/2} \).
In the 1950’s, the Hungarian sociologist Sandor Szalai studied friendship relationships between children.

He observed that in any group of around 20 children he was able to find four children who were mutual friends, or four children such that no two of them were friends.

Is this an interesting sociological phenomena?