Recall: Plane Graphs

- A plane graph is a drawing of a graph in the plane such that the edges are non-crossing curves.
Recall: Planar Graphs

- The drawing on the left is not a plane graph. However, on the right we have a different drawing of the same graph, which is a plane graph.
- An abstract graph that can be drawn as a plane graph is called a planar graph.

Non-Planar Graphs

- Recall. We proved that $K_5$ and $K_{3,3}$ are not planar.
  - Thus, every graph that contains $K_5$ or $K_{3,3}$ as a subgraph is also not planar.
  - Are there graphs that do not contain $K_5$ and $K_{3,3}$ as subgraphs, and are not planar?
    - Yes, and we can use $K_5$ and $K_{3,3}$ to generate them.
More Non-Planar Graphs

- **Subdividing edges** of \( K_5 \) or \( K_{3,3} \) cannot make them planar.
  - If we have a plane drawing after the subdivision, the same drawing works for the original graph.

Reminder: Topological Minors

- A graph \( H \) is a *topological minor* of a graph \( G \) if \( G \) contains a subdivision of \( H \) as a subgraph.
Kuratowski's Theorem

- **Theorem.** A graph is planar if and only if it does not have $K_5$ and $K_{3,3}$ as topological minors.
  - We know that if a graph contains $K_5$ or $K_{3,3}$ as a topological minor, then it is not planar.
  - It remains to prove that every non-planar graph contains such a topological minor.

Kazimierz Kuratowski

Minimal Non-planar Graph

- A **minimal non-planar graph** is a non-planar graph $G$ such that any proper subgraph of $G$ is planar.
- What minimal non-planar graphs can you think of?
  - $K_5$ and $K_{3,3}$. 

![Diagram of minimal non-planar graphs](image)
Kuratowski Subgraphs

- Given a graph $G$, a Kuratowski subgraph of $G$ is a subgraph that is a subdivision of $K_5$ or of $K_{3,3}$.

Proof Strategy

- To prove Kuratowski's theorem, we need to prove that every non-planar graph contains a Kuratowski subgraph.
  - It suffices to prove this only for minimal non-planar graphs.
- Strategy:
  - Show that every minimal non-planar graph with no Kuratowski subgraph must be 3-connected.
  - Then show that every 3-connected graph with no Kuratowski subgraph is planar.
Choosing the Unbounded Face

- **Lemma.** Let $G$ be a planar graph, and let $F$ be a set of edges that form the **boundary of a face** in an embedding of $G$. Then there exists a non-crossing drawing of $G$ where $F$ is the **boundary of the unbounded face**.

Proof

- We draw the graph on a sphere, and then **project it from a point on the face** $f$.
  - In the projection on the plane, $f$ will be the unbounded face.
Bad Math Joke #1

- Q: What do you call a young eigensheep?
- A: A lamb, duh!

Connectedness of Minimal Non-planar Graphs

- **Claim.** A minimal non-planar graph must be 1-connected.
  - Assume for contradiction that there exists a minimal non-planar graph $G$ that is not connected.
  - Let $C$ be one connected component of $G$.
  - By the minimality of $G$, both $C$ and $G - C$ are planar.
  - But then we can draw $C$ and then draw $G - C$ inside one of its faces. **Contradiction!**
2-Connectedness

- **Claim.** A minimal non-planar graph must **2-connected**.
  - Assume for contradiction that there exists a minimal non-planar graph $G = (V, E)$ that is not 2-connected.
  - There exists a vertex $v$ whose removal disconnects $G$.
  - Let $C$ be a component of $G - v$.
  - By the minimality of $G$, the induced subgraph on $C \cup \{v\}$ and $(V \setminus C) \cup \{v\}$ are both planar.
  - We can embed both graphs with $v$ on the unbounded face, and merge both copies of $v$.

**Illustration**
Preparing for 3-Connectedness

- **Claim.** Let $G \in (V, E)$ be a non-planar graph and let $x, y \in V$, such that $G - \{x, y\}$ is disconnected. Then there is a component $C$ of $G - \{x, y\}$ such that the induced subgraph on $C \cup \{x, y\}$ with the edge $(x, y)$ is non-planar.

![Graph Diagram]

**Proof**

- $C_1, \ldots, C_k$ – the components of $G - \{x, y\}$.
- $G_i'$ – the induced subgraph on $C_i \cup \{x, y\}$, plus the edge $(x, y)$.
- Assume **for contradiction** that $G_1', \ldots, G_k'$ are all planar.
  - $H_1$ – a plane drawing of $G_1'$.
  - $H_i$ (for $2 \leq i \leq k$) – drawing $G_i'$ (without crossings) in a face of $H_{i-1}$ with $(x, y)$ on its boundary, and merging the two copies of $x, y$.
  - Each $H_i$ is planar, including $H_k = G$.

  **Contradiction!**
Bad Math Joke #2

• **Q:** What do you get when you cross a mountain goat and a mountain climber?
• **A:** Nothing—you can’t cross two scalars.

3-Connectedness

• **Lemma.** Let \( G = (V, E) \) be a graph with fewest edges among all non-planar graphs without a Kuratowski subgraph. Then \( G \) is 3-connected.

• **Proof.**
  ◦ \( G \) is obviously a minimal non-planar graph.
  ◦ By a previous lemma, \( G \) is 2-connected.
  ◦ We need to prove that there are no vertices \( x, y \in V \) such that \( G - \{x, y\} \) is disconnected.
Proof

- Assume for contradiction that there exist \( x, y \in V \) such that \( G - \{x, y\} \) is disconnected.
  - \( C_1, \ldots, C_k \): the components of \( G - \{x, y\} \).
  - By the previous lemma, there exists \( C_i \) such that the induced subgraph on \( C_i \cup \{x, y\} \) plus the edge \((x, y)\) is non-planar. Denote it as \( H \).
  - By the minimality of \( G \), \( H \) contains a Kuratowski subgraph \( K \).
  - Since \( G \) does not contain \( K \), it must be that \((x, y) \in K\) and \((x, y) \notin E\).

Proof (cont.)

- Let \( C' \) be another connected component of \( G - \{x, y\} \).
- In \( G \) there is a path \( P \) between \( x \) and \( y \) that uses only vertices of \( C' \).
- Combining \( P \) with the other edges of \( K \) yields a Kuratowski subgraph of \( G \). **Contradiction!**
Recap

• We proved that a smallest non-planar graph without a Kuratowski subgraph is 3-connected.
  ◦ To complete the proof of Kuratowski’s Theorem, we prove that every 3-connected graph without a Kuratowski subgraph is planar.

Bad Math Joke #3

• Q: What do you get if you cross an elephant and a banana?
• A: $|\text{elephant}| \cdot |\text{banana}| \cdot \sin \theta$. 
Contraction Cannot Generate Kuratowski Subgraphs

- **Lemma.** Let $G = (V, E)$ be a graph with no Kuratowski subgraph. Then **contracting any edge** $e \in E$ does not result in a Kuratowski subgraph.

- **Proof.**
  - $G_e$ – the graph that is obtained by contracting $e = (x, y)$ in $G$.
  - Assume **for contradiction** that $G_e$ contains a Kuratowski subgraph $H$.

Proof

- $v_e$ – the vertex in $G_e$ that is obtained by contracting $e = (x, y)$.
- If $v_e$ is not in $H$, then $H$ is also a subgraph of $G$. **Contradiction!**
- $v_e$ cannot have degree zero or one in $H$.
- If $v_e$ has degree two in $H$, we can find $H$ in $G$ by replacing $v_e$ with $x$ and/or $y$. **Contradiction!**
Proof (cont.)

- Consider the case where $v_e$ has degree $d_v \geq 3$ in $H$, and after expanding $e$ back $x$ (or $y$) is of degree $\geq d_v$ in $G$.
  - Then $H$ is also in $G$ with $x$ replacing $v_e$ and $y$ being a subdivision vertex. **Contradiction!**

- A single case remains: $H$ is a subdivision of $K_5$ and after expanding $e$ back both $x$ and $y$ are of degree 3.
  - In this case $G$ contains $K_{3,3}$. **Contradiction!**
  - In the figure, we have $y, a, b$ on one side and $x, c, d$ on the other.
Contractions and 3-Connectivity

- **Lemma.** Let $G = (V, E)$ be a 3-connected graph with $|V| \geq 5$. Then there exists an edge $e \in E$ whose contraction results in a 3-connected graph.

Proof

- Assume for contradiction that there exists a 3-connected $G = (V, E)$ with $|V| \geq 5$, such that contracting any $e \in E$ yields a graph $G_e$ that is not 3-connected.
  - For any $e \in E$, let $v_e$ denote the vertex of $G_e$ to which $e$ is contracted.
  - Since $G_e$ is not 3-connected, there exists $z_e \in V$ such that $G_e - \{v_e, z_e\}$ is disconnected.
Proof (cont.)

- Every \( e = (x, y) \in E \) has \( z_e \in V \) such that:
  - \( G_e - \{v_e, z_e\} \) is disconnected.
  - \( G - \{x, y, z_e\} \) is disconnected.

- We choose an edge \( e = (x, y) \) so that the size of the largest component \( C \) of \( G - \{x, y, z_e\} \) is maximized.
- Another component of \( G - \{x, y, z_e\} \).
- There must be an edge \( f \) between \( z_e \) and a vertex \( u \in C' \).

Proof (cont.)

- Let \( C' \) be another component. There is an edge \( f \) between \( z_e \) and a vertex \( u \in C' \).
  - By definition, \( G - \{z_e, u, z_f\} \) is disconnected.
  - The induced subgraph of \( C \cup \{x, y\} \) is connected. Also, deleting \( z_f \) from this subgraph cannot disconnect it, since this would imply that \( G - \{z_e, z_f\} \) is disconnected (but \( G \) is 3-connected!).
  - So \( G - \{z_e, u, z_f\} \) is disconnected and contains a component larger than \( C \).

Contradiction!
The End

- A physicist and a mathematician are sitting in a faculty lounge. Suddenly, the coffee machine catches on fire. The physicist grabs a bucket and leaps toward the sink, fills the bucket with water, and puts out the fire.

- Another day, and the same two sit in the same lounge. Again the coffee machine catches on fire. This time, the mathematician stands up, gets a bucket, and hands the bucket to the physicist, thus reducing the problem to a previously solved one.