Recall: $k$-connected Graphs

- A graph $G = (V, E)$ is said to be $k$-connected if $|V| > k$ and we cannot obtain a non-connected graph by removing $k - 1$ vertices from $V$ (together with their adjacent edges).

- Is the graph in the figure
  - 1-connected? Yes.
  - 2-connected? Yes.
  - 3-connected? No!
Recall: Connectivity

- Which graphs are 1-connected?
  - These are the connected graphs ($|V| > 1$).
- The connectivity of a graph $G$ is the maximum $k$ such that $G$ is $k$-connected.
- What is the connectivity of the complete graph $K_n$? $n - 1$.
- The graph in the figure has a connectivity of 2.

1- and 2-Connected Graphs

- We characterized all of the graphs that are 1-connected.
  - These are exactly the connected graphs.
- Can we characterize all of the graphs that are 2-connected?
  - What is the simplest type of 2-connected graphs? Cycles.
**G-paths**

- Given a graph $G$, a **G-path** is a path that meets $G$ only at its endpoints.

**2-connected Graphs**

- **Theorem.** A graph is 2-connected if and only if it can be constructed by repeatedly adding $G$-paths to a cycle.

- **Proof (easy direction).**
  - If a graph was built by repeatedly adding $G$-paths to a cycle, it cannot be disconnected by removing one vertex.
The Other Direction

- Assume **for contradiction** that a 2-connected graph $G = (V, E)$ cannot be obtained by repeatedly adding $C$-paths to a cycle $C$.

- **There is a cycle $C$ in $G$.**
  - Otherwise, $G$ is a spanning tree, and is obviously not 2-connected.

- **We repeatedly add $C$-paths** to $C$ using edges of $G$, until no such paths remain.
  - By definition, we obtain a subgraph $G' \subset G$.

Completing the Proof

- $G$ – 2-connected graph that cannot be obtained by adding $C$-paths to a cycle $C$.
- $G' \subset G$ – a **maximal** subgraph that can be obtained by adding $C$-paths to cycle $C$.
  - **Since $G$ is connected**, there is a vertex $v \in V$ that is connected by an edge to a vertex of $G'$.
  - **Since $G$ is 2-connected**, there must be another path between $v$ and $G'$.
  - **Contradicting the maximality of $G'$.**
Blocks

- **Recall.** Any graph can be decomposed into maximal connected components.

- A **block** is a maximal subgraph that is 2-connected.
  - Can we decompose every graph into blocks?

* The correct definition is 3 slides ahead.

Block Properties

- Can two blocks share a vertex? **Yes**

- Can two blocks share two vertices? **No**
  - Let $B_1, B_2$ be two blocks with at least two common vertices.
  - If we remove a vertex of $B_1$ from $B_1 \cup B_2$, by definition $B_1$ remains connected, and it also remains connected to $B_2$.
  - We cannot disconnect $B_1 \cup B_2$ by removing one vertex, so it is one big block.
The Decomposition

- **We decompose a graph into blocks.** Does every edge belong to block? **No**
  - We refer to edges between blocks as **bridges**.
  - We **extend the definition of a block** so that a bridge is also considered as a block.

The Accurate Definition of a Block

- A **block** is a maximal subgraph that cannot be disconnected by removing one vertex.

* We did not define whether an isolated vertex is a block, and this is just a matter of definition.
**st-disconnecting Set**

- Consider a graph $G = (V, E)$ and $s, t \in V$.
  - An **st-disconnecting set** is a subset $S \subset V \setminus \{s, t\}$ whose removal disconnects $G$, such that $s$ and $t$ are in different components.

![Graph Illustration](image)

**Menger’s Theorem**

- **Theorem (1927).** Consider a graph $G = (V, E)$ and vertices $s, t \in V$ such that $(s, t) \notin E$. Then the size of the smallest **st-disconnecting set** equals to the maximum number of **vertex-disjoint** paths between $s$ and $t$.

![Graph Illustration](image)
Proof

- \(k_{\text{path}}\) – maximum number of vertex disjoint paths.
- \(k_{\text{disc}}\) – minimum size of an \(st\)-disconnecting set.

We have \(k_{\text{disc}} \geq k_{\text{path}}\) since every \(st\)-disconnecting set must contain a vertex from every path.

We prove \(k_{\text{disc}} \leq k_{\text{path}}\) by induction on \(|V|\).

- **Induction basis.** When \(|V| = 2\), we have \(k_{\text{disc}} = k_{\text{path}} = 0\)

Induction Step

- \(N(s)\) – the set of neighbors of \(s\) in \(G\).
  - Notice that \(N(s)\) disconnects \(s\) from \(t\), and so does \(N(t)\).

- We partition the analysis of the induction step into two cases:
  - There exists a minimum-sized \(st\)-disconnecting set \(D\) such that \(D \neq N(s)\) and \(D \neq N(t)\).
  - Every minimum-sized \(st\)-disconnecting set is either \(N(s)\) or \(N(t)\) (one of these two sets might not be minimal).
The First Case

- Assume that there exists a minimum-sized $st$-disconnecting set $D$ such that $D \neq N(s)$ and $D \neq N(t)$.
  - Removing $D$ disconnects $G$ into several components.
  - $C_s$ - the component containing $s$.
  - $C_t$ - the component containing $t$.
  - How can we use the induction hypothesis?

The First Case (cont.)

- $G_s$ - the induced graph on $C_s \cup D$.
  - We add a vertex $t'$ to $G_s$ and edges between $t'$ and every vertex of $D$.
  - Since $D$ is an $st$-disconnecting set of $G$, it is also an $st'$-disconnecting set of $G_s$.
  - By the hypothesis, there are $|D| = k_{\text{disc}}$ vertex-disjoint paths from $s$ to $t'$.
Completing the First Case

- We have a set of vertex disjoint paths from $s$ to each of the $k_{disc}$ vertices of $D$.
- Similarly, we have a set of vertex disjoint paths from each of the $k_{disc}$ vertices of $D$ to $t$.
  - Combining the two yields a set of $k_{disc}$ vertex disjoint paths from $s$ to $t$.
  - That is, $k_{disc} \leq k_{path}$, completing the proof in this case.

The Second Case

- Assume that every minimum-sized $st$-disconnecting set is either $N(s)$ or $N(t)$.
  - That is, $v \in V \setminus (\{s, t\} \cup N(s) \cup N(t))$ is not in any minimum-sized $st$-disconnecting set.
  - By removing such a vertex $v$, we obtain a graph $G'$, also with a minimum $st$-disconnecting set of size $k_{disc}$.
  - By the hypothesis, $G'$ contains $k_{disc}$ vertex-disjoint paths between $s$ and $t$, and these are also vertex disjoint paths in $G$. 

A Missing Case

- What is still missing in case 2?
  - What if there is no vertex \( v \in V \setminus (\{s, t\} \cup N(s) \cup N(t)) \)?
  - Let \( C = N(s) \cap N(t) \), \( N_s = N(s) \setminus C \), and \( N_t = N(t) \setminus C \).
  - Any disconnecting set must contain \( C \), which also corresponds to \( |C| \) paths of the form \( s \to v \to t \), where \( v \in C \).
  - Each of the other \( k_{\text{disc}} - |C| \) vertices of the minimum disconnecting set is either in \( N_s \) or \( N_t \).

Completing the Missing Case

- Let \( C = N(s) \cap N(t) \), \( N_s = N(s) \setminus C \), and \( N_t = N(t) \setminus C \). Consider the bipartite subgraph on \( N_s \cup N_t \) (removing edges between vertices of the same side).
- The minimum disconnecting set contains \( k_{\text{disc}} - |C| \) vertices in this subgraph. These vertices form a **minimum vertex cover**.
Recall: König’s Theorem

- **Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Then the size of a maximum matching of $G$ is equal to the size of a minimum vertex cover of $G$.

Using Vertex Covers

- Since $G'$ has a minimum vertex cover of size $k_{\text{disc}} - |C|$, it has a matching $A$ of the same size.
  - Each matching edge corresponds to a path $s \rightarrow v \rightarrow u \rightarrow t$ ($v \in N_s$ and $u \in N_t$).
  - These paths are vertex disjoint, so we again have at least $k_{\text{disc}}$ vertex-disjoint paths.
Conclusions

• **Menger’s theorem** yields an alternative definition of $k$-connectedness.
  - **Original definition.** A graph $G = (V, E)$ is said to be $k$-connected if $|V| > k$ and we cannot obtain a non-connected graph by removing $k - 1$ vertices from $V$.
  - **Equivalent definition.** A graph $G = (V, E)$ is said to be $k$-connected if $|V| > k$ and between any two vertices $s, t \in V$ there are at least $k$ vertex-disjoint paths.

Verifying $k$-Connectivity

• **Problem.** Given a graph $G = (V, E)$ and an integer $k > 0$, describe an algorithm for checking whether $G$ is $k$-connected.
Solution

- For every pair of vertices $s, t \in V$, we check whether there are $\geq k$ vertex-disjoint paths between $s$ and $t$.
  - $G$ is $k$-connected if and only if all of the $\binom{|V|}{2}$ checks pass.
- How can we check whether there are $k$ vertex-disjoint paths between $s$ and $t$?
  - We did this in 6a using flow networks.

Building a Flow Network

- A quick reminder from 6a:
  - The source is $s$. The sink is $t$.
  - The capacities are all 1.
  - We split every edge into a pair of anti parallel edges.
  - **We split every $v \in V$ into $v_{in}$ and $v_{out}$**.
More Efficient

- We showed how to check whether a graph is $k$-connected by finding maximum flow in $|V| \choose 2$ flow networks.
- By more involved argument, it suffices to find $|V| - 1$ maximum flows.

The End

ANSWER: natural selection

WAIT! THAG CALCULATES SHORTEST DISTANCE TO LOCATION MAXIMIZING PROBABILITY OF SURVIVAL.

THAG MAKE FIRE.

THAG INVENT WHEEL.

NOW THAG WILL...

SABERTOOTH!

$L = \int_c^x \sqrt{1 + y'^2} \, dx$

$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x} \frac{\partial F}{\partial y'} = 0$

QUESTION: Why are there so many more jocks than nerds in the world today?