Ma/CS 6b
Class 2: Matchings

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There aren’t enough crocodiles in the presentations

Why won’t you tell me how to solve the homework?!

Only today! 75% off for Morphine and Xanax.

Could you open every class by playing Flight of the Valkyries?
National Resident Matching Program

- Every medical student who is about to graduate ranks hospitals in which she wants to do her residency.

- Every hospital ranks students that it is interested in.

- Every year, over 20,000 applicants apply to about 1,800 programs.

- How can we handle this?

Bipartite Graphs

- A graph $G = (V, E)$ is bipartite if we can partition $V$ into disjoint subsets $V_1, V_2 \subset V$ such that every edge of $E$ is between a vertex of $V_1$ and $V_2$.

- Equivalently, the vertices of $V$ can be colored red and blue such that no edge is monochromatic.
A Useful Graph Family

• $K_{m,n}$ – a **complete bipartite graph** with $m$ vertices on one side and $n$ on the other (there is an edge between every two vertices on opposite sides).

$K_{2,5}$

Reminder: Matchings

• A **matching** in an graph is a set of vertex-disjoint edges.
• The **size** of a matching is the number of edges in it.
• A **maximum matching** of $G$ is a matching of maximum size.
Reminder: Perfect Matchings

- A perfect matching of a graph $G = (V, E)$ is a matching of size $|V|/2$.

Back to the Medical Students

- How can we approach our medical students problem?
  - Bipartite graph – a vertex in $V_1$ for each student. A vertex in $V_2$ for each hospital.
  - An edge exists between a student and a hospital if they are interested in each other.
Solving the Problem?

- What should we do with the student-hospital graph?
  - We can find the maximum matching, but there are two problems with this.

First Problem

- **Problem.** Some hospitals might wish to take more than one resident.
- **Solution.** (as we saw in 6a)
  - If a hospital wants to take \( k \) residents, in the graph we have \( k \) vertices for it.
Second Problem

- **Problem.** We did not consider the rankings of the students and hospitals.
  - We might have chosen the red matching.
  - However, perhaps student $A$ prefers hospital $\beta$, student $B$ prefers hospital $\alpha$, and similarly for the hospitals.

Alternating Paths

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a matching of $G$.
- A path is *alternating* for $M$ if it starts with an unmatched vertex of $V_1$ and every other edge of it is in $M$. 
Augmenting Paths

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a matching of $G$.
- A path is augmenting for $M$ if it is an alternating path of $M$, and it ends in an unmatched vertex.

Using Augmenting Paths

- Consider a matching $M$ and an augmenting path $P$ of $M$.
- By switching in $P$ the edges that are in $M$ with the edges that are not, we obtain a larger matching.
Augmenting Paths and Matchings

• **Claim.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph and let $M$ be a matching in $G$. Then $M$ is not a maximum matching iff there exists an augmenting path for it.

• **Proof.**
  ◦ If there is an augmenting path, we can use it to find a larger matching, so $M$ is not a maximum matching.
  ◦ It remains to prove that if $M$ is not a maximum matching, there is an augmenting path for it.

Completing the Proof

• Let $M^*$ be a maximum matching of $G$.
• Let $F$ be the set of edges that are either in $M$ or in $M^*$, but not in both. Set $G' = (V, F)$.
• In $G'$, every vertex is of degree at most two.
• Thus, $G'$ is composed of paths, cycles, and isolated vertices. Since $|M| < |M^*|$, there must be at least one augmenting path for $M$. 

[Diagram of paths and cycles]
Traffic Cameras

- **Problem.** The city of Pasadena wants to have traffic cameras that cover all of the roads of the city.
  - A camera covers 360° and sees far enough to cover a road at least until the next intersection.
  - How can we efficiently find the minimum number of cameras that are necessary?

Considering the Problem as a Graph

- We build a graph:
  - A vertex for every intersection.
  - An edge between every two adjacent intersections.

- What do we need to find in the graph?
  - A minimum set of vertices $S$ such that every edge is adjacent to at least one vertex of $S$. 
Vertex Covers

- Let $G = (V, E)$ be a graph. A vertex cover of $G$ is a set of vertices $V' \subseteq V$ such that every edge of $E$ is adjacent to at least one vertex of $V'$.

More About Vertex Covers

- No polynomial-time algorithm is known for finding the minimum vertex cover.
- This is a main open problem in theoretical computer science.
  - Significantly easier in bipartite graphs.
König’s Theorem

**Theorem.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Then the size of a maximum matching of $G$ is equal to the size of a minimum vertex cover of $G$.

**Proof.**
- $M$ – a maximum matching.
- $C$ – a minimum vertex cover.
- Since the edges of $M$ are vertex-disjoint and $C$ must contain a vertex of each, we have $|C| \geq |M|$.

**Proof (cont.)**
- $C$ – a minimum vertex cover.
- $M$ – a maximum matching.
- We saw that $C$ is larger or equal than $M$.
- To complete the proof, it suffices to find a vertex cover of size $|M|$.
- We build a subset $V' \subseteq V$ by taking one vertex out of each edge $e = (a, b)$ of $M$.
  - If an alternating path of $M$ ends in $b$, then we add $b$ to $V'$.
  - Otherwise, we add $a$ to $V'$. 
Proof (cont.)

- $V'$ consists of one vertex of each edge $(a, b) \in M$.
  - If an alternating path ends in $b \in M$ we add $b$ to $V'$.
  - Otherwise, we add $a$ to $V'$.
- Assume for contradiction that there is an edge $(a, b) \in E$ that is not covered by $V'$.
  - Either $a$ or $b$ must be matched in $M$, since otherwise $M$ is not a maximum matching.

The Case where $b$ is Matched

- Assume that $b$ is matched in $M$, but not $a$.
  - Then $(a', b) \in M$ for some $a' \in V_1$.
  - Since $b \notin V'$, we have $a' \in V'$ and no alternating path ends at $b$.
  - But $(a, b)$ is such an alternating path! Contradiction!
The Case where $a$ is Matched

- Assume that $a$ is matched in $M$.
  - Then $(a, b') \in M$ for some $b' \in V_2$.
  - Since $a \notin V'$, we have $b' \in V'$ and there is an alternating path $P$ ending at $b'$.
  - If $(a, b') \notin P$, then the path $P + (a, b') + (a, b)$ is an alternating path ending in $b$.
    Since $b$ is unmatched, this an augmenting path for $M$, contradicting the maximality of $M$.

Illustration #1

- If $(a, b') \notin P$, then the path $P + (a, b') + (a, b)$ is an alternating path ending in $b$. 

\[ \begin{array}{c}
  p_0 & \rightarrow & q_1 \\
  p_5 & \rightarrow & q_7 \\
  p_{12} & \rightarrow & b' \\
  a & \rightarrow & b
\end{array} \]
Illustration #2

- It cannot be that \((a, b') \in P\). No alternating path can end in \((a, b')\).
  - In the path, we move from \(V_1\) to \(V_2\) only with unmatched edges.

Concluding the Proof

- \(M\) – a maximum matching.
- We defined a subset \(V' \subset V\) of size \(|M|\) and proved that it is a vertex cover.
- We also proved that any vertex cover is of size at least \(|M|\), implying that \(V'\) is a minimum vertex cover.
  - That is, the minimum vertex cover has the same size as the maximum matching.
Vertex Covers in Bipartite Graphs

- **Problem.** Describe an efficient algorithm for finding a vertex cover in a bipartite graph $G = (V_1 \cup V_2, E)$.

- **Solution.**
  - From 6a, we know an algorithm for finding a maximum matching $M$ in a bipartite graph.
  - We pick one vertex out of each edge $(a, b) \in M$. If an alternating path ends in $b$ we pick $b$. Otherwise, we pick $a$.
  - But how do we know whether such a path exists?

Finding an Alternating Path

- Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, and let $M$ be a maximum matching.

- We wish to find whether there is an alternating path for $M$ ending at a vertex $b \in V_2$.
  - We run a variant of BFS from $b$.
  - We already did this in detail in 6a.
The End

There's a certain type of brain that's easily disengaged. If you show it an interesting problem, it immediately drops everything else to work on it.

This has led me to invent a new sport: NERD SNIPING. See that physicist crossing the road?

Hey!

On this infinite grid of ideal one-ohm resistors, what's the equivalent resistance between the two marked nodes?

It... Hmm... Interesting. Maybe if you start with... no, wait... Hmm... you could...

Fooooom

I will have no part in this. Ohm. Lore. A saw. It's fun. Physics is two parts, mathematicians three.