Ma/CS 6b
Class 1: Graph Recap

By Adam Sheffer

Course Details

• Instructor: Adam Sheffer.
• TA: Cosmin Pohoata.
• 1pm Mondays, Wednesdays, and Fridays.
• http://math.caltech.edu/~2015-16/2term/ma006b/
• Main book: Introduction to Graph Theory, 2nd edition, Douglas West.
  ◦ Also Graph Theory, 4th edition, Reinhard Diestel.
Course Structure

- Grade:
  - 70% homework assignments.
  - 30% final exam.
  - No midterm

- Six assignments:
  - Due by noon on Tuesdays.
  - Please read the homework policy on the website!

Topics

- More advanced graph topics:
  - Planar graphs.
  - Ramsey theory.
  - Extremal graph theory.
  - Spectral graph theory.

- A few other topics:
  - The probabilistic method.
  - Error correcting codes.
Graphs

- **Undirected graph**
- **Directed graph**

- In this class, unless stated otherwise, the graph is undirected.

Graph Representation

- We write $G = (V, E)$. That is, the graph $G$ has **vertex set** $V$ and **edge set** $E$.

- **Example.** In the figure:
  - $V = \{a, b, c, d, e\}$.
  - $E = \{(a, b), (a, d), (a, e), (b, c), (b, e)\}$. 
Graph Representation (cont.)

- $V = \{a, b, c, d, e\}$.
- $E = \{(a, b), (a, d), (a, e), (b, c), (b, e)\}$.

Simple Graphs

- An edge is a **loop** if both of its endpoints are the same vertex.
- Two edges are **parallel** if they are between the same pair of vertices.
- A graph is **simple** if it contains no loops and no parallel edges.
- **Unless stated otherwise, the graph is simple.**
Degrees

• The degree of a vertex is the number of edges that are adjacent to it.

• Claim. In any graph, the sum of the degrees of the vertices is even.

• Proof. Every edge contributes 1 to the degree of exactly two vertices. Thus,

\[ \sum_{v \in V} \deg(v) = \sum_{e \in E} 2 = 2|E|. \]

Subgraphs and Average Degree

• Given two graphs \( G = (V, E) \) and \( G' = (V', E') \). We say that \( G' \) is a subgraph of \( G \) if \( V' \subseteq V \) and \( E' \subseteq E \).
  
  ◦ We say that \( G' \) is an induced subgraph on \( V' \subseteq V \) if \( E' \) contains exactly the edges of \( E \) that connect two vertices of \( V' \).

• The average degree of a graph \( G = (V, E) \) is

\[ \deg(G) = \frac{1}{|V|} \sum_{v \in V} \deg(v) = \frac{2|E|}{|V|}. \]
Induced Subgraphs

Which of these is an induced subgraph?

Subgraphs with Large Degrees

Claim. For every graph $G = (V, E)$ with $|E| \geq 1$ there exists an induced subgraph $H$ of $G$, such that the minimum degree in $H$ is larger than $\text{deg}(G)/2$. 

$|E|/|V| = 7/6$
Solution

- Set \( k = \text{deg}(G) / 2 = |E|/|V| \).

- **We repeatedly remove vertices of degree at most \( k \) until none are left.**
  - If no vertex of \( G \) has degree \( \leq k \), we are done.
  - **Otherwise, how can we make sure that we do not remove the entire graph?**
    - Denote our sequence of subgraphs as \( G = G_0, G_1, G_2, ..., G_m \).
    - At each step, we remove one vertex and at most \( k \) edges. We thus have \( \text{deg}(G_m) \geq \text{deg}(G_{m-1}) \geq \cdots \geq \text{deg}(G_0) \geq 2k \).

Solution (cont.)

- Denote our sequence of subgraphs as \( G = G_0, G_1, G_2, ..., G_m \).

- At each step, we remove one vertex and at most \( k \) edges. We thus have \( \text{deg}(G_m) \geq \text{deg}(G_{m-1}) \geq \cdots \geq \text{deg}(G_1) > 2k \).

- **Since \( \text{deg}(G_m) \geq 2k \), it must contain edges.**
Paths and Cycles

- **Path** between $a$ and $b$.

- **Cycle** through $a$.

A cycle is a path that starts and ends in the same vertex.

More on Paths and Cycles

- A path/cycle is said to be **simple** if it does not visit any vertex more than once.

- The **length** of a path/cycle is the number of edges that it consists of.

- **Example.** A simple cycle of length 5.
Connected Graphs

• A graph $G = (V, E)$ is **connected** if for any pair $u, v \in V$, there is a path in $G$ between $u$ and $v$.

<Insert Diagram Here>

Connected Components

• A **connected component** (for short, **component**) of a graph $G = (V, E)$ is a maximal connected subgraph of $G$.
  ◦ For example, a graph with **three** connected components:

<Insert Diagram Here>
Paths and Degrees

• **Problem.** Let $G = (V, E)$ be a graph such that the degree of every $v \in V$ is at least $d$ (for some $d \geq 2$). Prove that $G$ contains a path of length $d$.

A graph with minimum degree 3.

Proof

• Assume **for contradiction** that a longest path $P$ is of length $c < d$.
• Consider a vertex $v$ which is an endpoint of $P$.
• Since $\deg v \geq d \geq c + 1$, it must be connected to at least one vertex $u \notin P$.
• By adding the edge $(v, u)$ to $P$, we obtain a longer path, **contradicting the maximality of $P$**.
Distance

- Consider an undirected graph $G = (V, E)$ and two vertices $v, u \in V$.
- The *distance* in $G$ between $u$ and $v$, denoted $d(u, v)$, is the length of the shortest path between $u$ and $v$.
- The *diameter* of $G$ is the maximum distance between two vertices of $G$.
  - That is, $\max_{u, v} d(u, v)$.

Diameter and Cycles

- **Claim.** Let $G = (V, E)$ be a graph of diameter $D$ that contains at least one cycle. Then $G$ contains a cycle of length at most $2D + 1$.

- **Example.**
  - Diameter: $2$
  - Length of shortest cycle: $5$
Proof

- Assume for contradiction that the length of the shortest cycle $C$ is at least $2D + 2$.
  - Consider two vertices $u, v$ with a distance of $D + 1$ in $C$.
  - If there is no shorter path between $u$ and $v$, we get a contradiction to the diameter being $D$.
  - If there is a shorter path between $u$ and $v$, we get a contradiction to $C$ being the shortest cycle in $G$.

More Degrees and Distances

- **Problem.** Consider a graph $G = (V, E)$ and integers $k, d \geq 3$, such that
  - The degree of every vertex of $V$ is at most $d$.
  - There exists a vertex $v \in V$ such that for every $u \in V$ we have $d[u, v] \leq k$.

What is the maximum number of vertices that $V$ can contain?
Solution

• We partition the vertices of $V$ according to their distance from $v$:
  ◦ How many vertices satisfy $d[v, u] = 0$? 1
  ◦ How many vertices satisfy $d[v, u] = 1$?
    • At most $d$.
  ◦ How many vertices satisfy $d[v, u] = 2$?
    • At most $d(d - 1)$.
  ◦ How many vertices satisfy $d[v, u] = i$?
    • At most $d(d - 1)^{i-1}$, for every $1 \leq i \leq k$.

Solution (cont.)

• We have the bound
  $$|V| \leq 1 + d + d(d - 1) + \cdots + d(d - 1)^{k-1}$$
  $$= 1 + d \frac{(d - 1)^k - 1}{(d - 1) - 1} = \frac{d(d - 1)^k - 2}{d - 2}.$$  
• Is this tight?
  ◦ Yes
Trees and Forests

- In a graph, a **tree** is a connected subgraph containing no cycles.
- A **forest** is a set of non-connected trees.

Leaves

- Given a tree $T$, a **leaf** of $T$ is a vertex of degree 1.
- **Claim.** Every tree contains a leaf.
- **Proof.** Consider a vertex $v$ in $T$.
  - If $v$ has degree 1, we are done.
  - Otherwise, we travel the tree without crossing any edge more than once.
  - No vertex is visited twice since there are no cycles in $T$. Thus, eventually we will get stuck.
  - The vertex that we got stuck in is of degree 1.
The Size of a Tree

- Given a tree with \( n \) vertices, how many edges are in it?
  - Exactly \( n - 1 \).
  - **Proof sketch.** By induction. By removing a leaf we obtain a tree by one vertex and one edge.

Rooted Trees

- A **rooted tree** is a tree with a special vertex – the **root** – that is singled out.
- We draw the tree with the root on top, and the edges “grow downwards”.
- A vertex \( v \) is the **parent** of a vertex \( u \) if there is an edge \( (u, v) \) and \( v \) is above \( u \).
  - Each vertex, except for the root, has a **unique parent**.

\[ s \text{ is the root and } t\text{'s parent} \]
The BFS Algorithm

The BFS algorithm receives a graph $G = (V, E)$ and a vertex $s \in V$.

- It outputs a BFS tree, containing shortest paths from $s$ to any vertex reachable from $s$.
- A rooted tree with root $s$.

Levels of the BFS Tree

- The $i$'th level of the BFS tree is the set of vertices $v \in V$ that satisfy $d(v) = i$.

* This is the origin of the name Breadth First Search.
The End

And over there we have the Labyrinth guards. One always lies, one always tells the truth, and one stabs people who ask tricky questions.