Problem 1 (10.1.8).

Proof. (a) We have to check that Tor(M) is an abelian subgroup of M and that for any \( r' \in R, m \in \text{Tor}(M) \) then \( r'm \in \text{Tor}(M) \).

Since \( 0 \in \text{Tor}(M) \), this is nonempty. If \( m, n \in \text{Tor}(M) \) then there exist \( r, s \in R \) with \( rm = sn = 0 \). Then \( rs(m - n) = 0 \), so \( m - n \in \text{Tor}(M) \) (note: \( rs \neq 0 \) - here is where we are using that \( R \) is an integral domain!) and \( \text{Tor}(M) \) is an abelian subgroup of \( M \). For any \( r' \in R \) we have \( r(r'm) = (rr')m = r'(rm) = 0 \), since integral domains are commutative; hence \( r'm \in \text{Tor}(M) \).

(b) Let \( R = \mathbb{Z}/6\mathbb{Z} \) viewed as a module \( M \) over itself. Then \( 2, 3 \in \text{Tor}(M) \) but \( 2 + 3 \notin \text{Tor}(M) \).

(c) Since \( R \) has zerodivisors, there exist nonzero \( a, b \in R \) with \( ab = 0 \). Then for any \( m \in M \), either \( bm = 0 \) which means \( m \in \text{Tor}(M) \), or \( a(bm) = 0 \) so \( bm \in \text{Tor}(M) \).

\[ \square \]

Problem 2 (10.2.8).

Proof. For any \( m \in M \), there exists \( r \in R \) with \( rm = 0 \). Then \( r\varphi(m) = \varphi(rm) = \varphi(0) = 0 \), so \( \varphi(m) \in \text{Tor}(N) \).

\[ \square \]

Problem 3 (10.3.5).

Proof. Let \( m_i, 1 \leq i \leq k \) be a set of generators for \( M \). Since \( M \) is torsion, there exist nonzero \( r_i \in R \) such that \( r_i m_i = 0 \) for all \( i \). Let \( r = r_1 \cdots r_k \). A general element \( m \) of \( M \) is a linear combination of the generators, say \( m = a_1 m_1 + \ldots + a_k m_k \) where \( a_i \in R \). Then \( r \cdot m = 0 \) and \( r \neq 0 \) since \( R \) is an integral domain. Thus \( r \) is a nonzero element of \( R \) annihilating all of \( M \), i.e. \( \text{Ann}(M) \) is nontrivial.

\[ \square \]

Problem 4 (10.3.18).

Proof. Let \( q_i = \prod_{j \neq i} p_i^{\alpha_j} \). Let’s first show that \( q_i M = M_i \). The inclusion \( q_i M \subseteq M_i \) is clear since \( p_i^{\alpha_i} \cdot (q_i m) = am = 0 \), hence \( q_i m \in M_i \). Conversely, let \( m \in M_i \). Since \( R \) is a PID, there exist elements \( r, s \) such that \( rq_i + sp_i^{\alpha_i} = 1 \). Then

\[ m = 1 \cdot m = (rq_i + sp_i^{\alpha_i}) = q_i rm \in q_i M. \]

Thus \( M_i = q_i M \).

Now we prove that \( M = \oplus M_i \). Let’s first show that \( M = M_1 + \ldots + M_k \). The inclusion \( \supset \) is clear. Conversely, let \( m \in M \). Since \( \gcd(q_1, \ldots, q_k) = 1 \), there are \( r_i \in R \) with \( \sum r_i q_i = 1 \). Then \( m = 1 \cdot m = \sum r_i \cdot q_i m \in M_1 + \ldots + M_k \).

Finally, let’s check that the above sum is direct. For that we show that \( M_i \cap (M_1 + \ldots + M_{i-1} + M_{i+1} + \ldots + M_k) \) is trivial (i.e. contains only 0). Let \( m \) be an element in the intersection. Then \( p_i^{\alpha_i} m = 0 \) and \( q_i m = 0 \). Writing again \( 1 = rq_i + sp_i^{\alpha_i} \), we get \( m = 1 \cdot m = 0 \).

\[ \square \]

Problem 5.

Proof. (1) Let \( m, n \in N[I] \). For every \( a \in I \) we have \( a(m + n) = am + an = 0 + 0 = 0 \), hence \( m + n \in N[i] \).
(2) Let \( m \) be a generator for \( M \). Then a homomorphism \( f : M \to N \) is uniquely determined by the image \( f(m) \). Note that \( af(m) = f(am) = 0 \) for all \( a \in \text{Ann}(m) \), so \( f(m) \) is indeed in \( N[\text{Ann}(m)] \). Define \( \varphi : \text{Hom}_R(M, N) \to N[\text{Ann}(m)] \) by \( \varphi(f) = f(m) \). The zero homomorphism goes to 0 and \( \varphi(f + g) = f(m) + g(m) = (f + g)(m) = \varphi(f + g) \).

For any \( n \in N[\text{Ann}(m)] \), we can define a homomorphism \( f : M \to N \) sending \( m \mapsto n \). This gives an inverse homomorphism for \( \varphi \), so \( \varphi \) is an isomorphism.

(3) If \( M \) is irreducible, then any nonzero element of \( M \) generates \( M \).

If \( f(m) = 0 \), then \( f \equiv 0 \). Assume \( f(m) \neq 0 \). We want to show \( f \) is injective, i.e. \( f(am) \neq 0 \) for any \( a \) such that \( am \neq 0 \), \( a \in R \). Assume that \( f(am) = af(m) = 0 \). Since \( am \neq 0 \) and it generates \( M \), there is \( b \in R \), such that \( bam = m \). Then \( f(m) = f(bam) = bf(am) = 0 \), contradiction to the assumption \( f(m) \neq 0 \).

(4) Let \( M, N \) be two irreducible modules and \( f : M \to N \) an \( R \)-linear map. Then \( \ker f \) and \( \text{im} f \) are submodules of \( M \) and \( N \) respectively. From irreducibility, \( \ker f = 0 \) or \( M \). The latter case means \( f \equiv 0 \). In the first case, \( f \) is injective. Then \( \text{im} f \) cannot be 0, so \( \text{im} f = N \), i.e. \( f \) is surjective. Thus \( f \) is bijective.

(5) \( 0 \in N[I] \) is clear. We saw that \( N[I] \) is closed under addition. Let now \( n \in N[I] \) and \( r \in R \). Then \( a(rn) = r(an) = r \cdot 0 = 0 \), for all \( a \in I \), so \( rn \in N[I] \). Thus \( N[I] \) is a submodule of \( N \).

(6) Remember that the abelian group homomorphism above sent \( f : M \to N \) to \( f(m) \). We just have to check that this is \( R \)-linear. Indeed, let \( r \in R \). Then \( rf \) is the homomorphism sending \( m \) to \( rf(m) \). Thus \( \varphi(rf) = (rf)(m) = rf(m) = r\varphi(m) \).

(7) A division algebra simply means a not necessarily commutative ring where every nonzero element is invertible (sort of a noncommutative field). By part (4) applied to \( N = M \), every nonzero element is an isomorphism, so it is invertible, hence \( \text{End}_R(M) \) is indeed a division algebra.

Now we have to check that (the field) \( R/m \) is in the center. For every element \( \bar{a} \) of \( F \) pick a representative class \( a \) in \( R \) (under \( R \to F = R/m \)). Then \( m \mapsto am \) is an endomorphism for every such \( a \). Call it \( f_a \).

Moreover for \( a \neq b \) distinct representatives, these endomorphisms are different, i.e. \( f_a \neq f_b \). If \( f : M \to M \) is any element of our division algebra, then

\[ f(f_a(m')) = f(am') = af(m') = f_a(f(m')) \]

for all \( m' \in M \). Thus \( f_a \) commutes with every \( f \in \text{End}_R(M) \). Thus \( \{f_a\} \cong F = R/M \) lies in the center of \( \text{End}_R(M) \).

\[ \square \]

**Problem 6.**

**Proof.** (1) We want to check \( (r, s) \circ m = rm \) for all \( m \in M \), makes \( M \) into an \( R \times S \)-module. We have

\[
(r, s) \circ (m_1 + m_2) = r(m_1 + m_2) = rm_1 + rm_2 = (r, s) \circ m_1 + (r, s) \circ m_2
\]

\[
(r_1 + r_2, s_1 + s_2) \circ m = (r_1 + r_2)m = r_1m + r_2m = (r_1, s_1) \circ m + (r_2, s_2) \circ m
\]

\[
(r_1r_2, s_1s_2) \circ m = (r_1r_2)m = (r_1)((r_2, s_2) \circ m) = (r_1, s_1) \circ ((r_2, s_2) \circ m),
\]

and finally \( (1, 1) \circ m = 1 \cdot m = m \).
(2) Let $T$ be an $R \times S$-module. Let $M = (1, 0)T$ and $N = (0, 1)T$. We show that $M$ is an $R$-module. By symmetry, $N$ will be an $S$-module then. We have that $(1, 0) \cdot 0 \in M$, hence $M$ is non-empty. Now consider $(1, 0)t, (1, 0)t' \in M$ and $(r, 0) \in R$. Then we have

$$(1, 0)t + (r, 0)(1, 0)t' = (1, 0)t + (r, 0)(r, 0)t' = (1, 0)(t + rt') \in M$$

Thus, $M$ is an $R$-module.

Now let’s check that $T = M \oplus N$. First, we need to show that $T = M + N$. Consider $t \in T$. We can write $t = (1, 1)t = (1, 0)t + (0, 1)t \in M + N$. Then we need to check that if $t \in M \cap N$, then $t = 0$. Indeed, suppose $t = (1, 0)x = (0, 1)y$. But we have that $x = (1, 1)x = (1, 0)x + (0, 1)x = (0, 1)(x+y)$. Multiplying on both sides by $(0, 1)$, we get $(0, 1)x = (0, 1)(x+y) = (0, 1)x + (0, 1)y$, which implies that $(0, 1)y = 0$. Thus $t = 0$, as wanted.