Due: Wednesday, February 24, 2016 at 1am.

All numbered problems are from Dummit and Foote, Third Ed.
All problems will be graded. Show all work to receive full credit.

Read section: 12.1 of the textbook (compare to section 5.2 of the textbook).

- From section 12.1: problems 2, 5,

- From section 12.1: problem 16. Also, (a) deduce that if $M$ is finitely generated and there exists a surjective $R$-morphism $M \to N$ then $N$ is finitely generated; (b) given an example of a submodule of a finitely generated module which is not finitely generated. (Hint: Let $R = \mathbb{Q}[x_1, x_2, \ldots x_n, \ldots]$ the polynomial ring in infinitely many variables with coefficients in $\mathbb{Q}$. Prove that the ideal $I = (x_1, \ldots x_n, \ldots)$ of $R$ generated by all variables is not finitely generated.)

- Assume $R$ is an ID. For $n \geq 1$, let $\phi : R^n \to R^n$ be a $R$-linear morphism, and write $A \in M_n(R)$ for the associated matrix. I.e., $\phi(x) = Ax$. Prove (1) $\phi$ is injective if and only if $\det(A) \neq 0$. (2) $\phi$ is bijective if and only if $\det(A)$ is a unit. (Hint: Let $F$ be the fraction field of $R$, consider $R^n \subset F^n$. Note that the map $\phi : R^n \to R^n$ extends to a unique map $\Phi : F^n \to F^n$. Recall the definition of the matrix $A^*$ adjoint to $A$. Note that $A^* \in M_n(R).$)

- Let $R$ be a commutative ID (not necessarily a PID), and $M$ an $R$-module. For any finite set $S$ of $M$, consider the $R$-linear morphism $\phi_S : R^{[S]} \to M$ which maps $(a_s) \mapsto \sum_{s \in S} a_s s$. The finite set $S$ is a basis of $M$ if $\phi_S$ is an isomorphism. The finite set $S$ is a generating $M$ if $\phi_S$ is surjective. The finite set $S$ is a linearly independent $M$ if $\phi_S$ is injective. Assume $M$ is free of rank $n$. Prove or give a counterexample: (1) Every basis of $M$ has size $n$. (2) Every set of linearly independent vectors has size at most $n$. (3) A set of linearly independent vectors of size $n$ is a basis. (4) Every generating set has size at least $n$. (5) A generating set of size $n$ is a basis. (6) A generating set consisting of linearly independent vector is a basis. (7) Every linearly independent set extends to a basis. (8) Every generating set contains a basis. (Hint: Let $F$ be the fraction field of $R$, there exists a $F$-vector space $V$ containing $M$, of dimension equal to the rank of $M$.)