Exercise 1. Yes, it’s possible to have three random variables but are dependent, but pairwise independent. Here is an example:

\begin{align*}
X &= \text{face of dice 1} \\
Y &= \text{face of dice 2} \\
Z &= X + Y \mod 6.
\end{align*}

We can check pairwise independence for each of the three pairs:

\begin{align*}
P(X = i, Y = j) &= P(X = i)P(Y = j) \quad \forall i, j \in \{1, \ldots, 6\} \\
P(X = i, Z = j) &= P(X = i, X + Y = j \mod 6) \\
&= P(X = i, Y = j - i \mod 6) \\
&= P(X = i)P(Y = j - i \mod 6) \\
&= P(X = i)P(Y = j \mod 6) \text{ by translation invariance of mod} \\
P(Y = i, Z = j) &= P(Y = i, X + Y = j \mod 6) \\
&= P(Y = i, Z = j - i \mod 6) \\
&= P(Y = i)P(X = j - i \mod 6) \\
&= P(Y = i)P(X = j \mod 6) \text{ by translation invariance of mod}
\end{align*}

However, \(X, Y, Z\) are dependent as can be seen from:

\begin{align*}
P(X = 0, Y = 0, Z = 0) &= P(X = 0, Y = 0, X + Y = 0) = \frac{1}{36} \\
P(X = 0)P(Y = 0)P(Z = 0) &= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{36} \neq \frac{1}{36}
\end{align*}

Exercise 2. Let \(X\) and \(Y\) be independent random variables, show that \(\text{Var}(X - Y) = \text{Var}(X + Y)\).

Lemma 1: If \(X\) and \(Y\) are independent, then \(\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)\).

\textbf{Proof}: this was proven in lecture. The key realization is that for independent \(X\) and \(Y\), we have that \(E(XY) = E(X)E(Y)\).
We obtain the desired result from
\[
\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = \text{Var}(X) + \text{Var}(-Y) = \text{Var}(X - Y)
\]
where the first equality and third equalities come from Lemma 1 and the second equality was shown in class but it also follows by writing down the definition of \( \text{Var} \) and using linearity of \( \mathbb{E} \).

**Exercise 3.** This problem is all done in a computing environment so see the attached image. Note that for \( P(|Z| \geq 2.58) \) I’m using the reflection symmetry of the normal distribution which gives: \( P(|Z| \geq 2.58) = 2P(Z \geq 2.58) = 0.00988 \).

**Figure 1.** Mathematica code for problem 3

```
In[80]:= T = {-3, -2, -1, 0, 1, 1.96, 2, 2.58, 3, 4, 5, 6}
Out[80]= {-3, -2, -1, 0, 1, 1.96, 2, 2.58, 3, 4, 5, 6}

In[87]:= CheckThisWorks = {Mean[T], Max[T]}
Out[87]= {1.62833, 6}

In[91]:= Prob1 = 1 - CDF[NormalDistribution[0, 1], 1.96]
Out[91]= 0.0249979

In[92]:= Prob2 = 1 - CDF[NormalDistribution[0, 1], 2.58]
Out[92]= 0.00494002

2 * Prob2
Out[93]= 0.00988003
```

**Exercise 4.** First, we will establish two lemmas

**Lemma 2:** For \( \mu = \mathbb{E}(X) \) we have that \( \text{Var}(X - \mu) = \text{Var}(X) \).
Proof: this comes from 6.5.1 (which is more general)

**Lemma 3:** For $Y$ with $E(Y) = 0$ we have

$$\text{std. dev} Y \geq E|Y|$$

**Proof:**

$$\text{std. dev. } Y = \sqrt{E(Y^2) - E(Y)^2} = \sqrt{E(Y^2 - 0^2)} = \sqrt{E(Y^2)} \geq E|Y|.$$  

where the inequality comes from realizing that $E(Y^2) = E(|Y|^2)$ and then applying Jensen’s inequality

$$f(E(X)) \leq E(f(X)).$$  

Jensen’s inequality holds for any random variable $X$ and convex function $f$. Here we use $X = |Y|$ and $f(x) = x^2$.

We obtain the result by setting $Y = X - \mu$ and realizing that Lemma 1 implies that

$$\text{std. dev. } X = \text{std. dev. } X - \mu := \text{std. dev. } Y$$

so that

$$\text{std. dev. } X \geq E|X - \mu|.$$

**Exercise 5.** An equivalent way to formulate this question is: select a permutation $\pi$ of the numbers $\{1, 2, \ldots, n\}$ uniformly at random. What is the expected number of fixed points of $\pi$, where $\pi(i) = i$?

Let $X_i$ be the indicator function for the event that $\pi(i) = i$. In other words, $X_i = 1$ when $\pi(i) = i$ and $X_i = 0$ otherwise. Let $X = \sum_{i=1}^{n} X_i$. We want to calculate $E(X)$. First observe that $E(X_i) = P(X_i) = 1/n$. Using this we obtain

$$E(X) = E \left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} (1/n) = 1$$

**Exercise 6.**

**Part 1:** Let $X$ be a Uniform[0,1]. We have

$$E[X] = \int_{0}^{1} x \, dx = \frac{1}{2}$$

$$E[X^2] = \int_{0}^{1} x^2 \, dx = \frac{1}{3}$$

so that the standard deviation is:

$$\text{std. dev. } X = \sqrt{E[X^2] - E[X]^2} = \sqrt{\frac{1}{3} - \frac{1}{4}} = \frac{1}{\sqrt{12}}$$
Part 2: The mean is: **0.494559**. The sample standard deviation is **0.287815**. These are VERY close to the values obtained above for the mean and standard deviation of Uniform[0,1].

![Mathematica code](image)

**Figure 2.** Mathematica code for problem 6 part 2

Part 3: See the plots in Figure 3.

Part 4: See the plot in Figure 4.

Part 4: The CDF makes it easier to check by eye if the data is uniform. This is because it “averages out” variations where bins either overshoot or undershoot the number of events expected per bin.
Figure 3. Histograms of bin width (default) and 0.02
Figure 4. Histogram of the CDF with bin width 0.02