Exercise 1.
The protocol consists of drawing 3 balls out of the urn (and recording their numbers). Here we do not
distinguish the order in which we draw the balls, so the outcome space can be modeled as
\[ \Omega = \{(a_1, a_2, a_3) \mid 1 \leq a_1 < a_2 < a_3 \leq 9\}. \]
The size of \( \Omega \) equals the number of ways of choosing 3 balls out of 9,
\[ \#\Omega = \binom{9}{3} = \frac{9!}{3! \cdot 6!} = 84. \]
Note that each outcome is equally likely.

Part 1.
We are interested in the event \( E = \{a_3 = 7\} \). We have to choose \( a_1 \) and \( a_2 \) from \( \{1, 2, 3, 4, 5, 6\} \), so
\[ \#E = \binom{6}{2} = 15. \]
This yields
\[ \text{Prob}(E) = \frac{\#E}{\#\Omega} = \frac{15}{84} = \frac{5}{28}. \]

Part 2.
We are interested in the event \( E = \{a_3 \geq 7\} \). Consider the complement \( E^c = \{a_3 \leq 6\} \). For the complement, we have to choose \( a_1, a_2 \) and \( a_3 \) from \( \{1, 2, 3, 4, 5, 6\} \), so \( \#E^c = \binom{6}{3} = 20 \). This yields
\[ \text{Prob}(E^c) = \frac{\#E^c}{\#\Omega} = \frac{20}{84} = \frac{5}{21}, \]
and consequently
\[ \text{Prob}(E) = 1 - \text{Prob}(E^c) = \frac{16}{21}. \]

Part 3.
For this part, \( E = \{a_2 = 4\} \). Then there are 3 choices for \( a_1 \) and 5 choices for \( a_3 \), so \( \#E = 5 \cdot 3 = 15 \).
This yields
\[ \text{Prob}(E) = \frac{\#E}{\#\Omega} = \frac{15}{84} = \frac{5}{28}. \]

Part 4.
Now we are interested in the event \( E = \{a_1 + a_2 + a_3 \text{ is even}\} \). For \( a_1 + a_2 + a_3 \) to be even, there are two possibilities: all three are even, or exactly one of them is even. There are four even numbers and five odd numbers. Hence we have \( \binom{4}{3} = 4 \) choices for the former and \( \binom{4}{1} \binom{5}{2} = 40 \) choices for the latter. These two cases are mutually exclusive, so \( \#E = 4 + 40 = 44 \). This yields
\[ \text{Prob}(E) = \frac{\#E}{\#\Omega} = \frac{44}{84} = \frac{11}{21}. \]

Remark. Even if we distinguish the order in which we draw the balls, we still get the same answer, because we will simply have to multiply both \( \#E \) and \( \#\Omega \) by 3!. \( \square \)
Exercise 2.

Part 1.
Assume that the universe is $1.4 \cdot 10^{10}$ years old, and the population of the earth is $7 \cdot 10^9$.
There are about $3 \cdot 10^7$ seconds per year, so humanity can create $(1.4 \cdot 10^{10}) \cdot (7 \cdot 10^9) \cdot (3 \cdot 10^2) \approx 3 \cdot 10^{27}$ arrangements during the age of the universe (so far).
At that rate it will take about $2.6 \cdot 10^{40}$ ages of the universe to go through all the possibilities.

Part 2.
Because you choose 13 cards out of 52, the number of distinct hands is \( \binom{52}{13} \approx 6.3 \cdot 10^{11} \).

Part 3.
The first player chooses 13 cards out of 52, the second player chooses 13 cards out of the remaining 39, the third player chooses 13 cards out of the remaining 26, and the fourth player gets the last cards.
Hence the number of distinct deals is \( \binom{52}{13} \binom{39}{13} \binom{26}{13} \binom{13}{13} \approx 1.3 \cdot 10^{23} \).

Exercise 3.

Part 1.
To lose in game A you must get zero aces.
The probability of not getting an ace for each throwing is \( \frac{5}{6} \), and the throwings are independent.
Therefore the probability of losing in game A is \( \left( \frac{5}{6} \right)^4 \approx 0.482253 \).

Part 2.
To lose in game B you must get zero or one ace.
The probability of getting zero aces is \( \left( \frac{5}{6} \right)^8 \approx 0.232568 \).
The probability of getting one ace is \( \binom{8}{1} \cdot \frac{1}{6} \cdot \left( \frac{5}{6} \right)^7 \approx 0.372109 \).
Since these two events are mutually exclusive, the probability of losing in game B is:
\[
\left( \frac{5}{6} \right)^{12} + \binom{12}{1} \cdot \frac{1}{6} \cdot \left( \frac{5}{6} \right)^{11} \approx 0.604677
\]

Part 3.
To lose in game C you must get zero, one, or two aces.
The probability of getting zero aces is \( \left( \frac{5}{6} \right)^{12} \approx 0.112157 \).
The probability of getting one ace is \( \binom{12}{1} \cdot \frac{1}{6} \cdot \left( \frac{5}{6} \right)^{11} \approx 0.269176 \).
The probability of getting two aces is \( \binom{12}{2} \cdot \binom{1}{1} \cdot \left( \frac{5}{6} \right)^{10} \approx 0.296094 \).
Since these three events are mutually exclusive, the probability of losing in game B is:
\[
\left( \frac{5}{6} \right)^{18} + \binom{18}{1} \cdot \frac{1}{6} \cdot \left( \frac{5}{6} \right)^{17} + \binom{18}{2} \cdot \frac{1}{6} \cdot \left( \frac{5}{6} \right)^{16} \approx 0.677426.
\]

Part 4.
By comparing the results from the previous parts, we conclude that game A has the smallest probability of losing.

\[\blacksquare\]
Optional Exercise.

For simplicity, we will write \( N = N(r,s) \). For \( i = 1, 2, \cdots, s \), let \( N_i \) be the number of runs of length \( r \) which starts on the \( i \)th toss. Then \( N = N_1 + N_2 + \cdots + N_{s-r+1} \) since \( N_i = 0 \) if \( i > s-r+1 \).

We easily see that \( N = 2 \) if \( s = r \) and \( N = 0 \) if \( s < r \). It remains to consider the case when \( s > r \).

Let us first consider the case when \( i = 1 \). Since the run starts at the first toss, the values for position 1 through \( r \) have one of the two values \( H \) or \( T \) while position \( r+1 \) must have the opposite value. Hence we have 2 choices for the first \( r+1 \) positions. There is no restriction for the remaining \( s-r-1 \) positions, meaning that we have \( 2^{s-r-1} \) choices for these positions. This gives \( N_1 = 2 \cdot 2^{s-r-1} = 2^{s-r} \).

Similarly, we get \( N_{s-r+1} = 2^{s-r} \).

Now we consider the case when \( 1 < i < s-r+1 \). The run starts at the \( i \)th toss, so the values for position \( i \) through \( i+r-1 \) have one of the two values \( H \) or \( T \) while position \( i-1 \) and position \( i+r \) must be of the opposite value. Hene we have 2 choices for these \( r+2 \) positions. The remaining \( s-r-2 \) positions are unrestricted, meaning that we have \( 2^{s-r-2} \) choices for these positions. This gives \( N_i = 2 \cdot 2^{s-r-2} = 2^{s-r-1} \).

Hence we deduce that \( N = 2 \cdot 2^{s-r} + (s-r-1)2^{s-r-1} = (s-r+3)2^{s-r-1} \) when \( s > r \).

The average number of runs of length \( r \) is given by

\[
\frac{N}{2^r} = \begin{cases} 
\frac{s-r+3}{2^{r+1}} & \text{if } s > r \\
\frac{1}{2^{r-1}} & \text{if } s = r \\
0 & \text{if } s < r
\end{cases}
\]