Solutions to Homework 6

(Note: some computation details omitted)

Problem 2.
(a) The matrix is upper triangular, so its eigenvalues are just the entries on its main diagonal: 2, 5, 4.
(b) The matrix is diagonalizable, as the eigenvalues are distinct (Corollary 4.2.3).
(c) We have eigenvectors \((1,0,0)^T\) corresponding to \(\lambda = 2\), \((2,1,0)^T\) corresponding to \(\lambda = 5\), and \((3,2,1)^T\) corresponding to \(\lambda = 4\). So

\[
\begin{pmatrix}
2 & 6 & -6 \\
0 & 5 & -2 \\
0 & 0 & 4 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
2 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 4 \\
\end{pmatrix} \begin{pmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{pmatrix}^{-1}
\]

Problem 3.
(a) Let \(\alpha\) be the basis \(\{1, x, x^2, x^3\}\) for \(\mathbb{P}^3\). Then

\[
[T]_\alpha = \begin{pmatrix}
0 & 0 & 2 & 0 \\
0 & 1 & 0 & 6 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
\end{pmatrix}
\]

Thus the eigenvalues of \(T\) are 0, 1, 2, and 3.
(b) Each eigenvalue here has an eigenspace of dimension 1, so it suffices to take an eigenvector for each value. With respect to \(\alpha\), we have \((1,0,0,0)^T\) for \(\lambda = 0\), \((0,1,0,0)^T\) for \(\lambda = 1\), \((1,0,1,0)^T\) for \(\lambda = 2\), and \((0,3,0,1)^T\) for \(\lambda = 3\). As polynomials, these are \(1, x, x^2 + 1, x^3 + 3x\), respectively.
(c) As determined in part (b), the basis \(\beta = \{1, x, x^2 + 1, x^3 + 3x\}\) works. We have \([T]_\beta = \text{diag}(0, 1, 2, 3)\).

Problem 4.
The given matrix happens to be diagonalizable. We have

\[
A = \begin{pmatrix}
4 & 5 \\
6 & 3 \\
\end{pmatrix} = \begin{pmatrix}
-5 & 1 \\
3 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 9 \\
\end{pmatrix} \begin{pmatrix}
-5 & 1 \\
3 & 1 \\
\end{pmatrix}^{-1}
\]

Now \(1^2 = 1\) and \(3^2 = 9\). Thus we can take

\[
B = \begin{pmatrix}
-5 & 1 \\
3 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 3 \\
\end{pmatrix} \begin{pmatrix}
-5 & 1 \\
3 & 1 \\
\end{pmatrix}^{-1} = \frac{1}{4} \begin{pmatrix}
7 & 5 \\
3 & 9 \\
\end{pmatrix}
\]

Since \(\text{diag}(1,3)^2 = \text{diag}(1, 9)\), it follows that \(B^2 = A\).

Problem 5.
(a) This is not positive-definite: \(((1,2)^T, (1,2)^T) = 1 - 4 = -3 < 0\).
(b) This is not linear. We have \((I, I) = \text{tr}(I + I) = 4, (2I, I) = \text{tr}(2I + I) = 6\), so \((2I, I) \neq 2(I, I)\).

Problem 6.
We have, via the definition of norm and using linearity of the inner product,

\[
\|x + y\|^2 + \|x - y\|^2 = (x + y, x + y) + (x - y, x - y)
= \|x\|^2 + \|y\|^2 + 2(x, y) + \|x\|^2 + \|y\|^2 - 2(x, y) = 2\|x\|^2 + 2\|y\|^2.
\]