Exercise 1. (5 pts) [Apostol Ch 2.12, Problem 3] A linear transformation $T : V_2 \rightarrow V_2$ maps the basis vectors $i$ and $j$ as follows:

$$T(i) = i + j, \quad T(j) = 2i - j.$$ 

(a) Compute $T(3i - 4j)$ and $T^2(3i - 4j)$ in terms of $i$ and $j$.

(b) Determine the matrix of $T$ and $T^2$.

(c) Solve part (b) if the basis $(i, j)$ is replaced by $(e_1, e_2)$, where $e_1 = i - j$, $e_2 = 3i + j$.

Solution. (a) We have $T(3i - 4j) = -5i + 7j$ and $T^2(3i - 4j) = 9i - 12j$.

(b) The matrix of $T$ with respect to $(i, j)$ is $egin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$. The matrix of $T^2$ with respect to $(i, j)$ is $egin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

(c) The matrix of $T$ with respect to $(e_1, e_2)$ is $egin{pmatrix} -7/4 & -1/4 \\ 1/4 & 7/4 \end{pmatrix}$. The matrix of $T^2$ with respect to $(e_1, e_2)$ is $egin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. □

Exercise 2. (5 pts) [Apostol Ch 2.12, Problem 10] Let $V$ and $W$ be linear spaces, each with dimension 2 and each with basis $(e_1, e_2)$. Let $T : V \rightarrow W$ be a linear transformation such that $T(e_1 + e_2) = 3e_1 + 9e_2$, $T^2(3e_1 + 2e_2) = 7e_1 + 23e_2$.

(a) Compute $T(e_2 - e_1)$ and determine the nullity and rank of $T$.

(b) Determine the matrix of $T$ relative to the given basis.

(c) Use the basis $(e_1, e_2)$ for $V$ and find a new basis of the form $(e_1 + ae_0, 2e_1 + be_2)$ for $W$, relative to which the matrix of $T$ will be in diagonal form.

Solution. (a) We have $T(e_2 - e_1) = e_1 - e_2$. $T$ has nullity 0 and rank 2.

(b) The matrix of $T$ with respect to $(e_1, e_2)$ is $egin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$.

(c) An appropriate basis for $W$ is $(e_1 + 5e_2, 2e_1 + 4e_2)$. □

Exercise 3. (5 pts) [Apostol Ch 2.20, Problem 9] Prove that the system

$$\begin{align*}
  x + y + 2z &= 2 \\
  2x - y + 3z &= 2 \\
  5x - y + az &= 6
\end{align*}$$

has a unique solution if $a \neq 8$. Find all solutions when $a = 8$. 

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Proof. To determine the solution of the system, use row reductions to reduce the associated augmented matrix into reduced row echelon form. Upon reducing, we get
\[
\begin{pmatrix}
1 & 1 & 2 & 2 \\
0 & 3 & 1 & 2 \\
0 & 0 & a-8 & 0
\end{pmatrix}.
\]
If \( a \neq 8 \), we can divide the third row by \( a-8 \) and continue reducing, obtaining the unique solution \((x, y, z) = (4/3, 2/3, 0)\). If \( a = 8 \), we continue reducing the above matrix to get solutions of the form \((x, y, z) = (4/3, 2/3, 0) + t(-5/3, -1/3, 1)\) for \( t \in \mathbb{R} \).

Exercise 4. (5 pts)[Apostol Ch 2.20, Problem 10]

(a) Determine all solutions of the system
\[
\begin{align*}
5x + 2y - 6z + 2u &= -1 \\
x - y + z - u &= -2
\end{align*}
\]

(b) Determine all solutions of the system
\[
\begin{align*}
5x + 2y - 6z + 2u &= -1 \\
x - y + z - u &= -2 \\
x + y + z &= 6.
\end{align*}
\]

Solution. To determine the solutions to the system, use row reductions to reduce the associated augmented matrix into reduced row echelon form.

(a) The solutions are given by
\[
\begin{pmatrix}
1 \\
6 \\
3 \\
0
\end{pmatrix} + a
\begin{pmatrix}
4 \\
11 \\
7 \\
0
\end{pmatrix} + b
\begin{pmatrix}
0 \\
-1 \\
0 \\
1
\end{pmatrix}
\]
where \( a, b \in \mathbb{R} \).

(b) The solutions are given by
\[
\begin{pmatrix}
3/11 \\
4/11 \\
19/11 \\
0
\end{pmatrix} + a
\begin{pmatrix}
4 \\
-11 \\
7 \\
22
\end{pmatrix}
\]
where \( a \in \mathbb{R} \).

Exercise 5. (5 pts)[Apostol Ch 2.20, Problem 13] Determine the inverse of the matrix
\[
\begin{pmatrix}
1 & 2 & 2 \\
2 & -1 & 1 \\
1 & 3 & 2
\end{pmatrix}.
\]

Solution. The easiest way to find the inverse is just to run the algorithm for obtaining the inverse matrix: consider the \(3 \times 6\) matrix with a the matrix above in
the left block and the $3 \times 3$ identity matrix in the right block and use row reduction until the left block is the identity. Its inverse is

\[
\begin{bmatrix}
\frac{-5}{3} & \frac{2}{3} & \frac{5}{3} \\
-1 & 0 & 1 \\
\frac{2}{3} & -\frac{1}{3} & -\frac{5}{3}
\end{bmatrix}
\]