Exercise 1. (5 pts) Consider the set $F = \{0, 1, \alpha\}$. Show that there is at most one way to give $F$ the structure of a field with 0 and 1 as the additive and multiplicative identities. (i.e. show that there is only one possible way to define addition and multiplication compatible with the field axioms. You do not need to show that the addition and multiplication you come up with satisfy all the axioms, since without some theory this would just be a very long list of boring checks.)

Proof. Let’s first define multiplication $\ast$. By the field axioms as well as their consequences, we have

\[
0 \ast a = a \ast 0 = 0 \text{ for all } a \in F,
\]

\[
1 \ast a = a \ast 1 = a \text{ for all } a \in F,
\]

so we only need to define $\alpha \ast \alpha$. Clearly $\alpha \ast \alpha = 0$ implies $\alpha = 0 \ast \alpha^{-1} = 0$, which is a not possible. Here $\alpha^{-1}$ is the multiplicative inverse of $\alpha$. Similarly, $\alpha \ast \alpha = \alpha$ implies $\alpha = 1$, which is again not possible. Thus we must have $\alpha \ast \alpha = 1$. To define the addition $+$, we again already have

\[
0 + a = a + 0 = a \text{ for all } a \in F
\]

so we’re left to define $1 + 1$, $1 + \alpha = \alpha + 1$ and $\alpha + \alpha$.

If $1 + \alpha = 1$, then adding $-1$ to both sides we get $\alpha = 0$, which is impossible. Similarly, $1 + \alpha = \alpha$ gives us the contradiction $1 = 0$. Thus we must have $1 + \alpha = 0$.

If $\alpha + \alpha = \alpha$, we get to the contradiction $\alpha = 0$. If $\alpha + \alpha = 0$, since $1 + \alpha = 0$, we get $\alpha = 1$, which is again a contradiction. Thus we must have $\alpha + \alpha = 1$.

If $1 + 1 = 1$, we get the contradiction $1 = 0$. If $1 + 1 = 0$, as we already know $1 + \alpha = 0$, we’ll have $\alpha = 1$, which is not possible. Thus we must have $1 + 1 = \alpha$.

As a result, we saw there is at most one way to define $+$ and $\ast$ such that the field structure of $F$ with 0 and 1 as additive and multiplicative identities is satisfied. $\square$

Exercise 2. (4 pts)[Apostol, I.5, Problem 28] Determine whether the set is a real linear space. For those are not, say which axioms fail to hold.

28. All vectors in $V_n$ that are linear combinations of two given vectors $A$ and $B$.

Proof. A vector $v \in V$ is a linear combination of $A, B$ iff it can be written as $v = \lambda A + \mu B$ for some $\lambda, \mu \in K$. Let

\[
U = \{v = \lambda A + \mu B \mid \lambda, \mu \in K\} \subset V
\]

be the subset of linear combinations of $A, B$. We have

\[
0 = 0 \cdot A + 0 \cdot B \in U,
\]

\[
v, v' \in V \implies v = \lambda A + \mu B, \quad v' = \lambda' A + \mu' B \quad \text{for some } \lambda, \mu, \lambda', \mu' \in K
\]

\[
\implies v + v' = (\lambda + \lambda') A + (\mu + \mu') B \in U,
\]

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\[ v = \lambda A + \mu B \in U, \quad \alpha \in K \quad \Rightarrow \quad \alpha v = (\alpha \lambda)A + (\alpha \mu)B \in U. \]

So, \( U \subseteq V \) is a subspace.

**Exercise 3.** (8 pts) [Apostol, I.10, Problem 22] In this exercise, \( L(S) \) denotes the subspace spanned by a subset \( S \) of a linear space \( V \). Prove each of the statements (a) through (f):

(a) \( S \subseteq L(S) \)

*Proof.* For all \( x \in S \), \( 1 \in F \) and \( x = 1 \cdot x \) by axiom 10. \( x \in L(S) \). \( \square \)

(b) If \( S \subseteq T \subseteq V \) and if \( T \) is a subspace of \( V \), then \( L(S) \subseteq L(T) \).

*Proof.* Suppose \( u \in L(S) \). There exist \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in S \), and \( c_1, \ldots, c_n \in F \) such that \( u = \sum_{i=0}^{n} c_i x_i \). As \( S \subseteq T \) and \( T \) is a \( F \)-vector subspace of \( V \), \( \sum_{i=0}^{n} c_i x_i \in T \). Hence \( u \in T \). \( \square \)

(c) A subset \( S \) of \( V \) is a subspace of \( V \) if and only if \( L(S) = S \).

*Proof.* \( (\Rightarrow) \) By part (a), \( S \subseteq L(S) \). \( S \) is a subspace and \( S \subseteq S \). By part (b), \( L(S) \subseteq S \).

(d) If \( S \subseteq T \subseteq V \), then \( L(S) \subseteq L(T) \).

*Proof.* By part (a), \( T \subseteq L(T) \). \( S \subseteq T \) implies \( S \subseteq L(T) \). By part (b), \( L(S) \subseteq L(T) \).

(e) If \( S \) and \( T \) are subspaces of \( V \), then so is \( S \cap T \).

*Proof.* Exercise 4. \( \square \)

(f) If \( S \) and \( T \) are subsets of \( V \), then \( L(S \cap T) \subseteq L(S) \cap L(T) \).

*Proof.* \( S \cap T \subseteq S \) and \( S \cap T \subseteq T \). By part (d), \( L(S \cap T) \subseteq L(S) \) and \( L(S \cap T) \subseteq L(T) \). Hence \( L(S \cap T) \subseteq L(S) \cap L(T) \). \( \square \)

(g) Give an example in which \( L(S \cap T) \neq L(S) \cap L(T) \).

*Proof.* Let \( V \) be the following \( \mathbb{R} \)-vector space: \( V = \mathbb{R} \), \( + \) is usual addition, \( 0 \) is the usual \( 0 \), and for each \( r \in \mathbb{R} \), and scalar multiplication by \( r \) is just usual multiplication by \( r \). Let \( S = \{1\} \) and \( T = \{2\} \). \( S \cap T = \emptyset \). \( L(S \cap T) = L(\emptyset) = \{0\} \). \( L(S) \cap L(T) = \mathbb{R} \cap \mathbb{R} = \mathbb{R} \). \( \square \)

**Exercise 4.** (5 pts) Let \( \mathcal{U} \) be a nonempty collection of subspaces of a vector space \( V \). Prove that \( W = \bigcap_{U \in \mathcal{U}} U \) is a subspace of \( V \).

*Proof.* To prove that \( W \) is a subspace, we must show that \( W \) is nonempty and that it is closed under addition and scalar multiplication.

Since \( \mathcal{U} \) is a nonempty collection of subspaces of \( V \) and \( \{0\} \) is an element of any subspace of \( V \), it follows that \( \{0\} \in U \) for any \( U \in \mathcal{U} \). Hence, \( \{0\} \in W \) and so \( W \) is nonempty.

Let \( x, y \in W \), so \( x, y \in U \) for all \( U \in \mathcal{U} \). Since \( x, y \in U \) and \( U \) is a subspace, we have \( x + y \in U \). Since \( x + y \in U \) for all \( U \in \mathcal{U} \), we have \( x + y \in W \) and so \( W \) is closed under addition.
Let \( x \in W \) and \( a \in \mathbb{R} \). Since \( x \in W \), we must have \( x \in U \) for all \( U \in \mathcal{U} \). Since \( U \) is a subspace, we have \( ax \in U \). Since \( ax \in U \) for all \( U \in \mathcal{U} \), it follows that \( ax \in W \) and so \( W \) is closed under scalar multiplication. \( \square \)

**Exercise 5.** (8 pts) Let \( \mathcal{F} \) be a system of \( m \) linear equations in \( n \) variables \( x_1, \ldots, x_n \)

\[
\sum_{j=1}^{n} a_{i,j} x_j = b_i, \quad 1 \leq i \leq m
\]

with \( a_{i,j}, b_i \in \mathbb{R} \) or \( \mathbb{C} \). A solution to \( \mathcal{F} \) is a vector \( v = (v_1, \ldots, v_n) \in V_n \) such that \( \sum_{j=1}^{n} a_{i,j} v_j = b_i \) for all \( i \). Determine for which \( (b_1, \ldots, b_m) \in V_m \) the set \( S(\mathcal{F}) \) of solutions of \( F \) is a subspace of \( V_n \). Show that your answer is correct.

**Proof.** The set \( S(\mathcal{F}) \) is a subspace of \( V_n \), so it must be closed under addition. Thus, if \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) are two solutions of \( \mathcal{F} \), \( x + y = (x_1 + y_1, \ldots, x_n + y_n) \) must also be a solution. Thus

\[
\sum_{j=1}^{n} a_{i,j} (x_j + y_j) = b_i, \quad 1 \leq i \leq m,
\]

so \( 2b_i = b_i \) for all \( 1 \leq i \leq m \). Thus \( (b_1, \ldots, b_m) = (0, \ldots, 0) \).

Let’s check that we get a subspace if \( (b_1, \ldots, b_m) = (0, \ldots, 0) \). For \( S(\mathcal{F}) \) to be a subspace, it must be non-empty and closed under addition and scalar multiplication. We already saw above that if \( (b_1, \ldots, b_m) = (0, \ldots, 0) \), then \( S(\mathcal{F}) \) is closed under addition. If \( \alpha \) a scalar, \( x = (x_1, \ldots, x_n) \in S(\mathcal{F}) \), then

\[
\sum_{j=1}^{n} a_{i,j} x_j = 0, \quad 1 \leq i \leq m,
\]

so

\[
\sum_{j=1}^{n} a_{i,j} \alpha x_j = 0, \quad 1 \leq i \leq m,
\]

and thus \( \alpha x \) is also in \( S(\mathcal{F}) \). Moreover, \( S(\mathcal{F}) \) is clearly non-empty since the zero vector in \( V_n \) is a solution when \( (b_1, \ldots, b_m) = 0 \). Thus, \( S(\mathcal{F}) \) is a subspace of \( V_n \) if and only if \( (b_1, \ldots, b_m) = 0 \). \( \square \)