1) Suppose that $G$ has a decomposition into three edge paths $P_1, \ldots, P_m$. Let $M$ be the set of middle edges in this decomposition. We claim this is a perfect matching. First of all, since the vertices connected by $M$ have degree two on the paths, any other edge will have degree 1, hence, could not be a middle edge, hence, this is a matching. Secondly, $|M| = k$, giving

$$|E(G)| = 3k,$$

and hence

$$\sum \deg v = 3|V(G)| = 2|E(G)| = 6k,$$

giving $k = \frac{1}{2}|V(G)|$, implying the matching is complete.

Conversely, suppose $M$ is a perfect matching, then $H = G \setminus M$, then $H$ is regular of degree 2. By the argument in class (but finite), it is a disjoint union of cycles. Assign a direction to each cycle, then for each $e \in M$, $e$ is incident to exactly two edges pointing to $e$ and two edges pointing away from $e$. Adjoining either those all edges pointing away from edges in $M$ or those edges pointing towards $M$ gives paths of length 3 in $G$.

2) Let us define a bipartite graph, $G = X \cup Y$, where $X = \{A_1, \ldots, A_m\}$ and $Y = \{B_1, \ldots, B_n\}$ where $\{A_i, B_j\}$ is an edge in $G$ if and only if $A_i \cap B_j \neq \emptyset$. Any renumbering of the $A_i$ corresponds to a matching in the bipartite graph. For a matching to exist, we require that property $H$ holds, that is to say for a subset $S \subseteq X$, $|\Gamma(S)| \geq |S|$. Let us take any collection of $k$ subsets, without loss of generality, $S = \{A_1, \ldots, A_k\}$, then if we let

$$A_S = \bigcup_{i=1}^{k} A_i,$$
	hen $|A_S| = kn$. Since the $B_j$'s also form a partition into subsets of size $n$, there are $\geq k$ subsets such that $B_j \cap A_S \neq \emptyset$, with equality holding only when a subset of the $B_j$'s form a partition of $A_S$. So we have a system of distinct representatives, we may simply relabel the $B$'s by the index of the set it represents.

3) We wish to argue that there exists a system of distinct representatives when there are more $n$-tuples in $A_1 \times \ldots \times A_n$ than $m$-tuples with at least two entries the same. The number of $n$-tuples is easily calculated as

$$M = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|.$$

Let $N$ be the number of $m$-tuples with at least two entries the same, $N$. If $N = M < 1$, then there is an SDR. We first consider $m$-tuples in which the $i$-th and $j$-th element are equal, $N_{ij}$. We have $|A_i \cap A_j|$ possible elements in which the $i$-th and $j$-th elements are equal, all other choices are free, giving that

$$N_{ij} = \frac{|A_i \cap A_j|M}{|A_i| \cdot |A_j|}.$$

When we sum, over $N_{ij}$ , we obtain an upper bound on $N$, since possibly some $m$-tuples are counted more than once, hence,

$$N < \sum_{1 \leq i < j \leq n} \frac{|A_i \cap A_j|M}{|A_i| \cdot |A_j|}.$$

Given this upper bound on elements with some entries the same, compared against the set of all $m$-tuples gives

$$N < \sum_{1 \leq i < j \leq n} \frac{|A_i \cap A_j|M}{|A_i| \cdot |A_j|} < M$$

which dividing by $M$ gives the required result.

4) Let us demonstrate by taking a number of steps, starting with $f = 0$, we first choose 3 special paths, which have been outlined in red, which the $\alpha$ values from top to bottom are 6, 6 and 4. This has made one edge saturated.
Iterating again, we get

Giving a maximal capacity of 20.

5) As in the original proof, the necessity of the theorem, we must show that if the condition “H” holds, then a perfect matching exists.

Let $G$ be the flow network formed from a bipartite graph $G = X \cup Y$ by directing all edges from $X$ to $Y$ and adding a source $\{s\}$ with edges $(s, x)$ for each $x \in X$ and a sink $\{t\}$ with edges $(y, t)$ for each $y \in Y$. The capacity on all edges to and from $\{s, t\}$ is 1 while the remaining edges have infinite (or $|X| + 1$) capacity.

The graph $G$ is assumed to satisfy the condition for every subset $A \subset X$ that $|\Gamma(A)| \geq |A|$ (i.e., condition $H$).

We claim $|X|$ is the max flow by showing the minimal cut contains $X \cup \{s\}$, it should be clear that the flow cannot exceed $|X|$, as $c(X \cup \{s\}, Y \cup \{t\}) = |X|$ forms an upper bound on flows. Consider the cut $(A = X' \cup Y' \cup \{s\}, B = (X \setminus X') \cup (Y \setminus Y') \cup \{t\})$. The resulting capacity includes every edge from $s$ to $X \setminus X'$ and from $\Gamma(X')$ to $\{t\}$. This means the capacity is

$$c(A, B) = (|X| - |X'|) + |\Gamma(X')| > |X|,$$

hence, the cut has a capacity greater than $|X|$, it follows the minimal capacity is $|X|$. Furthermore, there is a flow achieving this with integer values, which must be 0 or 1 by construction, which corresponds to a complete matching.

6) Let us count the number of SDR’s, $(x_1, \ldots, x_n)$. We have 2 cases; if $x_1 = 1$, then $(x_1, \ldots, x_n)$ is an SDR for the $A_i$, if and only if $(x_2, \ldots, x_n)$ is an SDR for the intersection of the $A_i$’s with $\{2, \ldots, n\}$, which is in one-to-one correspondence with an SDR for $[n-1]$, hence, if $x_1 = 1$, we have $S_{n-1}$ such SDRs.

If $x_1 = 2$, then this forces $x_2 = 1$, otherwise 1 is not chosen, which is impossible. Now $(x_3, \ldots, x_n)$ must be an SDR for the intersection of the $A_i$ with $\{3, \ldots, n\}$, which is in one-to-one correspondence with the SDRs of $[n-2]$. Hence, we have $S_{n-2}$ SDR’s in which $x_1 = 2$.

The sum of the above situations shows us that

$$S_n = S_{n-1} + S_{n-2},$$

in which case, this is a linear difference equation with constant coefficients. The particular solutions are of the form $S_n = \lambda^n$, where $\lambda$ solves

$$\lambda^2 - \lambda - 1 = 0,$$

which is solved by

$$\lambda_1 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}.$$  

The general solution is

$$S_n = A\lambda_1^n + B\lambda_2^n$$
where we have two initial values, $S_0 = S_1 = 1$. This shows $A = -B \neq 0$ and hence

$$\lim_{n} S_n^{1/n} = \lambda_1$$

7) Let $A$ be the $|V(D)| \times |E(D)|$ incidence matrix of a connected digraph. Since every edge points to a vertex and away from another vertex, the sum of the rows is 0, hence the rank of $A$ is less than $|V(D)|$. If some number less than $|V(D)|$ of the rows of $A$ are linearly dependent, then this would correspond disconnected components, contradicting the fact that $D$ is connected, hence, the rank of $A$ is $|V(D)| - 1$.

By the definition, a circulations can be identified with the vectors in the null space of $A$, i.e., the dimension of the kernel of $A$, which by the above calculation is given by

$$|E(D)| - \text{rank}(A) = |E(D)| - |V(D)| + 1.$$

Note: If you were to use the hint of the spanning tree, we could also identify a linearly independent set of circulations on each topological loop, which is counted by the number of edges not in the spanning tree (i.e., the dimension of the first Homology group) and hence arrive at the same answer.