Homework 2 : Ma121a - Combinatorics

Homework is due on Friday the 23rd of October at 12:00 noon. While collaboration is encouraged, you must write your own solutions.

1) Show that if the set \([9] = \{1, \ldots, 9\}\) is colored red and blue then there is a monochromatic arithmetic progression of length three. Provide an example of a coloring of \([8] = \{1, \ldots, 8\}\) with no monochromatic arithmetic progression of length three. (3 marks)

2) (Problem 3F, Page 34) Prove that for all \(r \in \mathbb{N}\), there is a minimal number \(N(r)\) with the following property. If \(n \geq N(r)\) and the integers \(\{1, \ldots, n\}\) are colored with \(r\) colors, then there are three elements, \(x, y\) and \(z\) (not necessarily distinct) with the same color and \(x + y = z\). Determine \(N(2)\) and that \(N(3) > 13\). (3 marks)

3) (Problem 3E, Page 33) A tournament on \(n\) vertices is an orientation of \(K_n\). A transitive tournament is a tournament for which the vertices can be numbered in a way that \((i, j)\) is an edge if and only if \(i < j\).
   (a) Show that if \(k \leq \log_2 n\), every tournament on \(n\) vertices has a transitive subtournament on \(k\) vertices. (4 marks)
   (b) Show that if \(k > 1 + 2 \log_2 n\), there exists a tournament on \(n\) vertices with no transitive subtournament on \(k\) vertices. (4 marks)

4) (Problem 3G, Page 34) Let \(m\) be given. Show that if \(n\) is large enough, every \(n \times n\) \((0, 1)\)-matrix has a principal submatrix of size \(m\), in which all the elements below the diagonal are the same, and all the elements above the diagonal are the same. (3 marks)

   For clarity, we present the definitions:
   - A \((0, 1)\)-matrix is a matrix whose entries are all either 0 or 1.
   - A principal submatrix of an \(n \times n\) matrix, \(A = (a_{ij})\) where \(i, j \in \{1, \ldots, n\}\) is a submatrix \(B\) of the form \(B = (a_{ij})\) where \(i, j \in I \subset \{1, \ldots, n\}\).

5) (Problem 3K, Page 34) Let \(G\) satisfy the conditions of Theorem 3.1 (minimal degree \(d\) with no copy of \(K_{d+1}\) as a subgraph). Show that by removing at most \(n/d\) edges we can find a subgraph, \(G'\), with chromatic number \(\leq d - 1\). (3 marks)