Math 108A Midterm Exam

1. (a) We check that $d$ satisfies each of the axioms for a metric.
To see that $d$ is symmetric, observe
\[ d((x, y), (x', y')) = \sqrt{d_1(x, x')^2 + d_2(y, y')^2} = \sqrt{d_1(x', x)^2 + d_2(y', y)^2} = d((x', y'), (x, y)), \]
where in the second equality we’ve used the symmetry of $d_1, d_2$. 
To see that $d$ is positive definite, note that $d((x, y), (x', y')) = 0$ means $\sqrt{d_1(x, x')^2 + d_2(y, y')^2} = 0$, which happens if and only if $d_1(x, x') = 0, d_2(y, y') = 0$, which by positive definiteness of $d_1, d_2$ happens if and only if $(x, y) = (x', y')$. 
To show that $d$ satisfies the triangle inequality, choose arbitrary $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$ and compute
\[
\begin{align*}
    d((x_1, y_1), (x_3, y_3)) &= \sqrt{d_1(x_1, x_3)^2 + d_2(y_1, y_3)^2} \\
    &\leq \sqrt{(d_1(x_1, x_2) + d_1(x_2, x_3))^2 + (d_2(y_1, y_2) + d_2(y_2, y_3))^2} \\
    &\leq \sqrt{d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2 + d_1(x_2, x_3)^2 + d_2(y_2, y_3)^2} \\
    &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)),
\end{align*}
\]
where in the second line we’ve used the triangle inequality for $d_1, d_2$ and in the third we’ve used the triangle inequality for the euclidean norm on $\mathbb{R}^2$.

(b) **Solution 1:** To show that $F_1 \times F_2$ is closed, it suffices to show that $(F_1 \times F_2)^c$ is open. To that end, let $(x, y)$ be an element of $(F_1 \times F_2)^c$. Then either $x \in F_1^c$ or $y \in F_2^c$. Assume without loss of generality that $x \in F_1^c$. Since $F_1$ is closed, $F_1^c$ is open, and thus there is an $r > 0$ for which the open ball $B_r(x)$ of radius $r$ centered at $x$ is contained in $F_1^c$. We claim $B_r((x, y)) \subset (F_1 \times F_2)^c$. To see this, let $(x', y')$ be an element of $B_r((x, y))$. Then
\[
d_1(x', x) \leq \sqrt{d_1(x', x)^2 + d_2(y', y)^2} = d((x', y'), (y', y')) < r,
\]
so $x' \in B_r(x) \subset F_1^c$. Thus, $(x', y') \in (F_1 \times F_2)^c$. So indeed, $B_r((x, y)) \subset (F_1 \times F_2)^c$. Since each point of $(F_1 \times F_2)^c$ has an open neighborhood contained in $(F_1 \times F_2)^c$, $(F_1 \times F_2)^c$ is open. Thus, $F_1 \times F_2$ is closed.

**Solution 2:** To show that $F_1 \times F_2$ is closed, it suffices to show that if $\{(x_n, y_n)\}_{n=1}^\infty$ is a sequence of points in $F_1 \times F_2$ convergent to a limit $(x, y)$ in $X$, then $(x, y)$ is in $F_1 \times F_2$. To this end, fix $\varepsilon > 0$ and find $N$ large enough that $d((x_n, y_n), (x, y)) < \varepsilon$ for all $n > N$. Then for all $n > N$, we have
\[
d_1(x_n, x) \leq \sqrt{d_1(x_n, x)^2 + d_2(y_n, y)^2} = d((x_n, y_n), (x, y)) < \varepsilon.
\]
Thus, $\{x_n\}_{n=1}^\infty$ converges to $x$. Since $F_1$ is closed, $x \in F_1$. By similar reasoning, $y \in F_2$. Thus, $(x, y) \in F_1 \times F_2$, which completes the proof.

2. We will define a sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements of $A$ by recursion as follows: first define $a_0 = x$.
Having defined $a_0, \ldots, a_n$, define $a_{n+1}$ to be an arbitrary element of $A \setminus \{a_0, \ldots, a_n\}$. This is possible since if $A \setminus \{a_0, \ldots, a_n\} = \emptyset$, then $A \subset \{a_0, \ldots, a_n\}$, showing that $A$ is finite, a contradiction. By
the definition, we have that \( a_i \neq a_j \) if \( i \neq j \).

Now define \( f : A \to A \setminus \{x\} \) by

\[
f(y) = \begin{cases} 
a_{n+1} & \text{if } y = a_n, 
y & \text{if } y \neq a_m \text{ for all } m \in \mathbb{N}. 
\end{cases}
\]

Since \( y \) can be equal to at most one \( a_n \) by our construction, \( f \) is well-defined. Since \( x = a_0, x \neq a_{n+1} \) for all \( n \in \mathbb{N} \), and so \( f \) indeed maps into \( A \setminus \{x\} \).

Suppose \( y \neq z \). Then if \( y, z \in \{a_n : n \in \mathbb{N}\} \), we must have \( y = a_i \) and \( z = a_j \) for \( i \neq j \), and so \( f(y) = a_{i+1} \neq a_{j+1} = f(z) \) since \( i + 1 \neq j + 1 \). If \( y \in \{a_n : n \in \mathbb{N}\}, z \notin \{a_n : n \in \mathbb{N}\} \), then \( f(y) = a_{i+1} \neq z = f(z) \). Finally, if \( y, z \notin \{a_n : n \in \mathbb{N}\} \), then \( f(y) = y \neq z = f(z) \). Hence \( f \) is injective.

Suppose \( b \in A \setminus \{x\} \). If \( b \in \{a_n : n \in \mathbb{N}\} \), then since \( b \neq x = a_0 \), we have \( b = a_n \) for \( n > 0 \). Hence \( f(a_{n-1}) = a_n = b \). Otherwise, if \( b \notin \{a_n : n \in \mathbb{N}\} \), then \( f(b) = b \). Hence \( f \) is surjective. We have shown that \( f : A \to A \setminus \{x\} \) is a bijection, and so \( A \) is equivalent to \( A \setminus \{x\} \).

3. Suppose \( x \in \overline{E} = E \cup E' \). If \( x \in E \), then \( x \in B_r(x) \cap E \) for every \( r > 0 \). If \( x \in E' \), then by definition, \( (B_r(x) \cap E) \setminus \{x\} \neq \emptyset \) for all \( r > 0 \). Hence in either case, we see that \( B_r(x) \cap E \neq \emptyset \) for all \( r > 0 \).

Now since \( \overline{E} \cap G \neq \emptyset \), fix an arbitrary \( x \in \overline{E} \cap G \). Since \( x \in G \) and \( G \) is open, there exists \( r > 0 \) such that \( B_r(x) \subseteq G \). By our result in the first paragraph, since \( x \in \overline{E} \), we have that \( B_r(x) \cap E \neq \emptyset \). Hence there exists \( y \in E \cap B_r(x) \subset E \cap G \), and so \( E \cap G \neq \emptyset \).

4. The desired result is true vacuously if \( Y \) is empty, so assume \( Y \neq \emptyset \). Then for each \( e \in E \), \( \{d(e, y) \mid y \in Y\} \) is a nonempty set of nonnegative real numbers and thus has a well-defined nonnegative infimum. For each \( e \in E \) and \( n = 1, 2, \ldots \), choose \( y_{e,n} \in Y \) with

\[
d(e, y_{e,n}) = \inf \{d(e, y) \mid y \in Y\} + \frac{1}{n}.
\]

Such a \( y_{e,n} \) exists, because if it didn’t, \( \inf \{d(e, y) \mid y \in Y\} + \frac{1}{n} \) would be a lower bound for \( \{d(e, y) \mid y \in Y\} \) greater than \( \inf \{d(e, y) \mid y \in Y\} + \frac{1}{n} \), which is absurd.

Now, let \( S = \bigcup_{n=1}^{\infty} \{y_{e,n} \mid e \in E\} \). Since \( E \) is at most countable, each \( \{y_{e,n} \mid e \in E\} \) is at most countable. Since a countable union of at most countable sets is at most countable, \( S \) is at most countable.

We claim \( S \) is dense in \( Y \). To see this, choose \( y \in Y \) and fix \( \varepsilon > 0 \). Choose an integer \( n > \frac{3}{\varepsilon} \). Since \( E \) is dense in \( X \), there is an \( e \in E \) with \( d(y, e) < \frac{\varepsilon}{3} \). We then have \( \inf \{d(e, y) \mid y \in Y\} \leq \frac{2\varepsilon}{3} \), so that

\[
d(e, y_{e,n}) \leq \frac{\varepsilon}{3} + \frac{1}{n} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.
\]

Now by the triangle inequality,

\[
d(y, y_{e,n}) \leq d(y, e) + d(e, y_{e,n}) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.
\]

Thus, every point of \( Y \) is in the closure of \( S \). So \( S \) is the desired at most countable dense subset of \( Y \).

5. By induction on \( k \in \mathbb{N} \), it follows that \( F_{n+k} \subseteq F_n \) for all \( n, k \in \mathbb{N} \), and so \( F_m \subseteq F_n \) for any \( m \geq n \).
Solution 1: By De Morgan’s laws, from $\bigcap_{n \in \mathbb{N}} F_n \subset G$, we deduce that $G^c \subset \bigcup_{n \in \mathbb{N}} F_n^c$. Since $G$ is open and each $F_n$ is closed, it follows that $G^c$ is closed and each $F_n^c$ is open, and so $\{F_n^c\}_{n \in \mathbb{N}}$ is an open cover of $G^c$. Since $X$ is compact and $G^c \subset X$ is closed, $G^c$ is compact, and so there exists a finite subcover of $G^c$: $G^c \subset F_{n_1}^c \cup \cdots \cup F_{n_k}^c$. By De Morgan’s laws again, we obtain $F_{n_1} \cap \cdots \cap F_{n_k} \subset G$. Letting $N = \max\{n_i : 1 \leq i \leq k\}$, we have $F_N \subset F_{n_1}$ for each $i$, and so $F_N = F_{n_1} \cap \cdots \cap F_{n_k} \subset G$. So for each $n \geq N$, we have $F_n \subset F_N \subset G$. Therefore, if $F_n \not\subset G$, then $n < N$, and so $F_n \subset G$ for all but finitely many $n \in \mathbb{N}$.

Solution 2: Assume, for sake of contradiction, that $F_n \not\subset G$ for infinitely many $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, there exists $m \geq n$ with $F_m \not\subset G$, and so if $F_n \not\subset G$, then $F_m \subset F_n \subset G$, a contradiction. This proves that our assumption implies that $F_n \not\subset G$ for all $n \in \mathbb{N}$. In particular, each $F_n \cap G^c \neq \emptyset$. Since $G$ is open, $G^c$ is closed, and so $\{F_n \cap G^c\}_{n \in \mathbb{N}}$ is a sequence of closed, non-empty subsets of $X$ with $F_{n+1} \cap G^c \subset F_n \cap G^c$, since $F_{n+1} \subset F_n$. Since $X$ is compact, each $F_n \cap G^c$ is also compact, so we can conclude by the finite intersection property that $\bigcap_{n \in \mathbb{N}} (F_n \cap G^c) \neq \emptyset$. So fix an $x \in \bigcap_{n \in \mathbb{N}} (F_n \cap G^c)$. Then $x \in \bigcap_{n \in \mathbb{N}} F_n$, but $x \not\in G$, contradicting that $\bigcap_{n \in \mathbb{N}} F_n \subset G$. So our initial assumption was false, and $F_n \subset G$ for all but finitely many $n \in \mathbb{N}$.