Solutions for Ma108a assignment 7, II

Problem 1

We define

\[ X = \left\{ (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty \right\} \]

\[ \|x\| = \sum_{n=1}^{\infty} |x_n| \quad \text{for } x = (x_n) \in X \]

To see that \( \| \cdot \| \) is a norm we note that

\[ \sum_{n} |x_n| \geq 0 \]

and that if the sum is 0 then every term is 0, i.e., \( x = 0 \). Also \( \| \lambda x \| = |\lambda| \cdot \|x\| \) for \( \lambda \in \mathbb{C} \). Finally the triangle inequality holds, since it holds in \( \mathbb{C} : \]

\[ \|x + y\| = \sum_{n} |x_n + y_n| \leq \sum_{n} (|x_n| + |y_n|) = \|x\| + \|y\| \]

Note that this also shows that \( X \) is a vector space.

To see that \( X \) is complete with respect to this norm suppose that \( (x^k) \) is Cauchy, and let \( \varepsilon > 0 \) be given. Find \( N \) so large that for \( k, \ell > N \) we have \( \|x^k - x^\ell\| < \varepsilon \). Then for any \( n \in \mathbb{N} \) we have

\[ |x^k_n - x^\ell_n| \leq \sum_{j=1}^{\infty} |x^k_j - x^\ell_j| = \|x^k - x^\ell\| < \varepsilon \]

So for each \( n \) the sequence \( (x^k_n)_{k\in\mathbb{N}} \) is Cauchy, and since \( \mathbb{C} \) is complete there is a limit sequence \( x = (x_n) \). We need to show that \( x \in X \) and that \( x^k \to x \) with respect to \( \| \cdot \| \). Let us handle these two issues separately (although, treading carefully, they can be handled simultaneously). So we show that

\[ \sum_{n=1}^{N} |x_n| \]

can be bounded independently of \( N \). Note that since \( (x^k) \) is Cauchy, the sequence \( \|x^k\| \) is bounded - let \( M \) be an upper bound. So we have

\[ \sum_{n=1}^{N} |x_n| \leq \sum_{n=1}^{N} |x_n - x^k_n| + \sum_{n=1}^{N} |x^k_n| \leq \sum_{n=1}^{N} |x_n - x^k_n| + M \]

Since \( x^k_n \to x_n \) for all \( n \) we can choose \( k \) so large that the first term is less than 1 (say). This shows that \( \sum_{n=1}^{N} |x_n| \leq M + 1 \) independently of \( N \), i.e., \( \|x\| \leq M + 1 \).

Now we show that \( \|x - x^k\| \) is small for large \( k \). So again fix \( N \in \mathbb{N} \) and consider

\[ \sum_{n=1}^{N} |x_n - x^k_n| \]
Let $\varepsilon > 0$ be given, and find $K \in \mathbb{N}$ so large that for $k, \ell \geq K$ we have $\|x^k - x^\ell\| < \varepsilon/2$. We claim that for $k \geq K$ the sum above is smaller than $\varepsilon$ independently of $N$. To see this note that since $x^k_n \to x_n$ for $k \to \infty$ for each $n$ we can find a $J > K$ (which may depend on $N$) so that

$$ \sum_{n=1}^{N} |x_n - x^J_n| < \varepsilon/2. $$

But since $J > K$ we have

$$ \sum_{n=1}^{N} |x_n - x^k_n| \leq \sum_{n=1}^{N} |x_n - x^J_n| + \sum_{n=1}^{N} |x^J_n - x^k_n| $$

$$ \leq \sum_{n=1}^{N} |x_n - x^J_n| + \sum_{n=1}^{\infty} |x^J_n - x^k_n| $$

$$ \leq \sum_{n=1}^{N} |x_n - x^J_n| + \|x^J - x^k\| $$

$$ < \varepsilon/2 + \varepsilon/2 $$

as desired. Note that we didn’t need $x^k \in X$ to perform the second step. So in fact we showed $x - x^k \in X$ with small norm. And since $x^k \in X$ and $X$ is a vector space, we get for free that $x \in X$.

**Problem 2**

We define $G : \mathbb{R}^2 \to \mathbb{R}$ by

$$ G(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases} $$

Clearly $G$ is continuous outside the origin. To see that $G$ is also continuous at $(0, 0)$ note that $|xy| \leq (x^2 + y^2)/2$, so

$$ |G(x, y)| \leq \sqrt{x^2 + y^2} \to 0 \text{ for } (x, y) \to (0, 0). $$

To see that the partial derivatives of $G$ exist, first fix $y_0 \in \mathbb{R}$. If $y_0 = 0$ then $G(x, y_0) = 0$ for all $x$, so the partial derivative is $\partial_x G(x, y_0) = 0$. If $y_0 \neq 0$ then we calculate

$$ \partial_x G(x, y_0) = \frac{y_0^3}{(x^2 + y_0^2)^3}, $$

A similar situation occurs for the partial derivative $\partial_y G(x_0, y)$. So for all $(x, y)$ both partial derivatives exist. But $G$ is not differentiable in the sense of Fréchet: To see this note that if it was, the differential $f'(0)$ would be the 2 by 1 matrix $(\partial_x G(0, 0), \partial_y G(0, 0)) = (0, 0)$, which by definition means that

$$ \frac{|G(x, y) - G(0, 0)|}{\|(x, y)\|} \to 0 \text{ for } (x, y) \to 0 $$

But note that for $x = y \neq 0$ we have

$$ \frac{|G(x, x)|}{\|(x, x)\|} = \frac{x^2}{2x^2} = 1/2. $$
Problem 3
This problem is word for word the same as the proof of Theorem 9.15 in Rudin. Note that we did not use any property of Euclidean space in the proof, so it carries over line for line. It is reproduced below for convenience (with some extra explanations).

Let $F : X \to Y$ be differentiable at $x_0$ and $G : Y \to Z$ be differentiable at $y_0 = F(x_0)$. Denote $A = F'(x_0)$ and $B = G'(y_0)$ and $H = G \circ F$. Now write

$$
\begin{align*}
&u(h) = F(x_0 + h) - F(x_0) - A(h) \\
&v(k) = G(y_0 + k) - F(y_0) - B(k).
\end{align*}
$$

By definition of differentiability we have

$$
\|u(h)\| = \varepsilon(h)\|h\|, \quad \|v(k)\| = \delta(k)\|k\|
$$

where $\varepsilon(h)$ and $\delta(k)$ are functions tending to 0 for $h, k \to 0$ respectively. So let $h \in X$ be given, and put $k = F(x_0 + h) - F(x_0)$. Then we have

$$
\|k\| = \|(A + \varepsilon(h))h\| \leq (\|A\| + \varepsilon(h))\|h\|. \quad (1)
$$

So we get

$$
egin{align*}
H(x_0 + h) - H(x_0) - BAh &= G(F(x_0 + h) - F(x_0) + F(x_0)) - G(F(x_0)) - BAh \\
&= G(y_0 + k) - G(y_0) - BAh \\
&= B(k - Ah) + v(k) \\
&= Bu(h) + v(k).
\end{align*}
$$

So we can calculate

$$
\begin{align*}
\|H(x_0 + h) - H(x_0) - BAh\| &= \|Bu(h) + v(k)\| \\
&\leq \|B\| \cdot \|u(h)\| + \|v(k)\| \\
&\leq \|B\| \cdot \|\varepsilon(h)\|\|h\| + \|v(k)\| \\
&\leq \left(\|B\| \cdot \|\varepsilon(h) + \delta(k)(\|A\| + \varepsilon(h))\|\right)\|h\|.
\end{align*}
$$

Now we consider the limit $h \to 0$ in $X$. By equation (1) we see that $k \to 0$ for $h \to 0$, and by definition this implies $\delta(k) \to 0$ for $h \to 0$. This shows that $H$ is differentiable at $x_0$ with derivative $BA$. 

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