LECTURE 6: MATRICES AND LINEAR ALGEBRA

Key Strategies.

(1) View determinants as polynomials.

(2) Interpret the characteristic polynomial as a recursion.

(3) Try to find some “meaningful” products.

Useful Facts.

(1) (Cayley-Hamilton Theorem) For a square matrix $A$, define $p(x) = \det(xI - A)$. Then $p(A) = 0$.

Remark. The polynomial $p(x)$ is called the characteristic polynomial of $A$.

(2) (Vandermonde determinant)

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Exercise. Prove the above formula for the Vandermonde determinant.

Problem 1. Calculate the determinant of the following $4 \times 4$ matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{pmatrix}.$$

Problem 2. Let $a$ be a real number. Evaluate the $n \times n$ determinant $A_n$ whose $(i, j)$th entry is $a^{i-j}$.

Problem 3. (2010 B6) Let $A$ be an $n \times n$ matrix of real numbers for some $n \leq 1$. For each positive integer $k$, let $A^{[k]}$ be the matrix obtained by raising each entry to the $k$th power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \cdots, n + 1$, then $A^k = A^{[k]}$ for all $k \leq 1$.

Problem 4. Let $A$ and $B$ be $n \times n$ matrices satisfying $A + B = AB$. Prove that $AB = BA$. 


Useful Properties of Determinants.

Let $A$ be an $n \times n$ matrix. Let $a_{i,j}$ denote the $(i,j)$th entry of $A$.

1. (Formula) Let $S_n$ be the set (or group) of all permutations of the set $\{1, 2, \ldots, n\}$.

$$
\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.
$$

2. (Inductive computation) Let $A_{(i,j)}$ denote the submatrix obtained from $A$ by removing the $i$th row and the $j$th column.

$$
\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{(i,j)}) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{(i,j)}).
$$

3. (Multi-linearity) As a function on column vectors, the determinant is multilinear. In other words,

$$
\det(v_1, v_2, \ldots, rv_i + sw, \ldots, v_n) = r \det(v_1, v_2, \ldots, v_i, \ldots, v_n) + s \det(v_1, v_2, \ldots, w, \ldots, v_n),
$$

for any $n$-dimensional vectors $v_1, v_2, \ldots, v_n, w$ and scalars $r, s$.

4. (Alternating on rows) The determinant changes the sign after interchanging two rows.

$$
\det(v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots, v_n) = - \det(v_1, v_2, \ldots, v_j, \ldots, v_i, \ldots, v_n).
$$

5. (Transpose) Let $A^T$ denote the transpose of $A$. Then $\det(A^T) = \det(A)$.

6. (Multiplicativity) If $B$ is another $n \times n$ matrix, $\det(AB) = \det(A) \det(B)$.

7. (Relationship with Singularity) The following are equivalent:

   (i) $A$ is invertible.
   (ii) The equation $Av = 0$ has no nontrivial solutions.
   (iii) The row vectors (or the column vectors) of $A$ are linearly independent.
   (iv) $\det(A) \neq 0$. 