LECTURE 3: GEOMETRY

Key Strategies.

(1) Work on the complex plane (or more generally, in \( \mathbb{R}^n \)).

(2) “Algebraize” given information.

(3) Look for symmetries (to simplify the problem).

Useful Facts.

(1) (Euler’s formula) \( v - e + f = 2 \) for polyhedrons or planar graphs. Here \( v, e, \) and \( f \) are respectively the number of vertices, edges, and faces.

(2) For a triangle on the complex plane with vertices \( z_0, z_1 = a + ib, z_2 = c + id \), the area is given by

\[
\frac{1}{4} |z_1 \bar{z}_2 - \bar{z}_1 z_2| = \frac{1}{2} |ad - bc| = \frac{1}{2} \left| \begin{array}{cc} a & c \\ b & d \end{array} \right|.
\]

Exercise. Suppose that four complex numbers \( z_1, z_2, z_3, \) and \( z_4 \) form a convex quadrilateral. Prove that the area of this quadrilateral is given by

\[
\frac{1}{2} (|z_3 - z_1|(|z_4 - z_2| - |z_3 - z_1|)(z_4 - z_2)).
\]

Problem 1. (2003 B5) Let \( A, B \) and \( C \) be equidistant points on the circumference of a circle of unit radius centered at \( O \), and let \( P \) be any point in the circle’s interior. Let \( a, b, c \) be the distance from \( P \) to \( A, B, C \), respectively. Show that there is a triangle with side lengths \( a, b, c \), and that the area of this triangle depends only on the distance from \( P \) to \( O \).

Problem 2. (2000 A3) The octagon \( P_1P_2P_3P_4P_5P_6P_7P_8 \) is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon \( P_1P_3P_5P_7 \) is a square of area 5, and the polygon \( P_2P_4P_6P_8 \) is a rectangle of area 4, find the maximum possible area of the octagon.

Problem 3. (2007 A6) A triangulation \( \mathcal{T} \) of a polygon \( P \) is a finite collection of triangles whose union is \( P \), and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side is a side of exactly one triangle in \( \mathcal{T} \). Say that \( \mathcal{T} \) is admissible if every internal vertex is shared by 6 or more triangles. Prove that there is an integer \( M_n \), depending only on \( n \), such that any admissible triangulation of a polygon \( P \) with \( n \) sides has at most \( M_n \) triangles.

Exercise. A triangulation of a polygon \( P \) is said to be regular if the number of edges emerging from each vertex is constant. Determine the maximum number of triangles in a regular triangulation of a given triangle.
Problem 4. (2000 A5) Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$. 