LECTURE 10: GRAPH THEORY

Key Strategies.

1. Look for the simplest formulation into the graph theory.
2. For bipartite graphs, try to figure out what a “matching” means.

Useful Facts.

1. For a graph $G = (V, E)$, $\sum_{v \in V} \deg(v) = 2|E|$.
2. A graph $G = (V, E)$ with $|E| \geq |V|$ contains a cycle.
3. (Hall’s Marriage Theorem) Let $G$ be a bipartite graph with bipartition $X, Y$. For any $W \subseteq X$, let $N(W)$ be the set of all “neighborhoods” of $W$, i.e., the set of all vertices in $Y$ connected to a vertex in $W$. Then there exists a matching between $X$ and $Y$ if and only if $|W| \leq |N(W)|$ for all $W \subseteq X$.

Problem 1. Consider a set of $2n - 1$ distinct points on a circle. Suppose that exactly $k$ of these points are to be colored black. Such a coloring is “good” if there is at least one pair of black points such that one of the arcs between them contains $n$ points (including themselves). Find the smallest value of $k$ so that every such coloring of $k$ points is good.

Problem 2. (2012 B3) A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the $n$ games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

Problem 3. Let $S = \{1, 2, \cdots, kn\}$, and suppose that $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$ are both partitions of $S$ into $n$ sets of size $k$. Prove that there exists a set $T$ of size $n$ such that $|T \cap A_i| = 1$ and $|T \cap B_i| = 1$ for all $i = 1, 2, \cdots, n$.

Problem 4. (2014 B3) Let $A$ be an $m \times n$ matrix with rational entries. Suppose that there are at least $m + n$ distinct prime numbers among the absolute values of the entries of $A$. Show that the rank of $A$ is at least 2.

Problem 5. The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.
**Problem 6.** A permutation matrix is an \( n \times n \) matrix which has exactly one entry of 1 in each row and each column and 0’s elsewhere. Let \( A \) be an \( n \times n \) matrix of nonnegative integers, in which each row and column sum up to the positive integer \( m \). Prove that \( A \) can be expressed as a sum of \( m \) permutation matrices.

**Remark.** Problem 2 of Homework 8 can be solved by using this result.

**Problem 7.** Let \( k \) and \( n \) be positive integers with \( k \leq n \). A Latin rectangle is a \( k \times n \) array filled with integers 1, 2, \( \cdots \), \( n \) such that entries in each row and column are distinct. Prove that every \( k \times n \) Latin rectangle can be extended to an \( n \times n \) Latin rectangle.