HOMEWORK 7 SOLUTIONS

Problem 1. (1991 B2) Suppose $f$ and $g$ are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers $x$ and $y$,

\[ f(x + y) = f(x)f(y) - g(x)g(y), \]
\[ g(x + y) = f(x)g(y) + g(x)f(y). \]

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x$.

Solution:

Substituting $x = y = 0$ in the two equations gives $f(0) = f(0)^2 - g(0)^2$ and $g(0) = 2f(0)g(0)$. The latter implies that either $g(0) = 0$ or $f(0) = \frac{1}{2}$. If $f(0) = \frac{1}{2}$, the former becomes $\frac{1}{2} = (\frac{1}{2})^2 - g(0)^2$, which is impossible since $g(0)$ is a real number. Hence we deduce that $g(0) = 0$. Now setting $y = 0$ in the original equation yields $f(x) = f(x)f(0) - g(x)g(0) = f(x)f(0)$. Since $f$ is non-constant, $f(0) = 1$.

Let $x$ be an arbitrary real number. For every real number $h$, we use the given equations to obtain

\[ \frac{f(x + h) - f(x)}{h} = \frac{f(x)f(h) - g(x)g(h) - f(x)}{h} = f(x) \cdot \frac{f(h) - 1}{h} - g(x) \cdot \frac{g(h)}{h}, \]
\[ \frac{g(x + h) - g(x)}{h} = \frac{f(x)g(h) + g(x)f(h) - g(x)}{h} = f(x) \cdot \frac{g(h)}{h} + g(x) \cdot \frac{(f(h) - 1)}{h}. \]

Since $f$ and $g$ are both differentiable, we take the limits to see

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f(x) \cdot \lim_{h \to 0} \frac{f(h) - 1}{h} - g(x) \cdot \lim_{h \to 0} \frac{g(h)}{h}, \quad (1) \]
\[ g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x) \cdot \lim_{h \to 0} \frac{g(h)}{h} + g(x) \cdot \lim_{h \to 0} \frac{(f(h) - 1)}{h}. \quad (2) \]

Observe that the limits on the right can be computed by

\[ \lim_{h \to 0} \frac{f(h) - 1}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0) = 0, \quad \lim_{h \to 0} \frac{g(h)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0). \]

Hence we can rewrite (1) and (2) as

\[ f'(x) = -g(x)g'(0), \quad g'(x) = f(x)g'(0). \quad (3) \]

Now define $h(x) = (f(x))^2 + (g(x))^2$. Note that $h$ is differentiable as both $f$ and $g$ are differentiable. Since $h(0) = (f(0))^2 + (g(0))^2 = 1$, it suffices to show that $h$ is constant, i.e., $h'(x) = 0$ for all $x$. This can
be easily seen by direct computation using (3):

\[ h'(x) = 2f(x)f'(x) + 2g(x)g'(x) = -2f(x)g(x)g'(0) + 2f(x)g(x)g'(0) = 0 \]

**Remark.** With some additional work, we can explicitly find \( f \) and \( g \). In fact, differentiating (3) gives

\[ f''(x) = -g'(0)f(x), \quad g''(x) = -g'(0)g(x). \]

Letting \( c = g'(0) \), this can be solved by \( f(x) = r \cos(cx) + s \sin(cx) \) and \( g(x) = r' \cos(cx) + s' \sin(cx) \) for \( r, r', s, s' \in \mathbb{R} \). Then by conditions \( f(0) = 1 \) and \( f'(0) = 0 \) we determine that \( f(x) = \cos(cx) \), and (3) gives \( g(x) = \sin(cx) \).

**Problem 2.** (2009 A2) Functions \( f, g, h \) are differentiable on some open interval around 0 and satisfy the equations and initial conditions

\[
\begin{align*}
  f' &= 2f^2g + \frac{1}{gh}, \quad f(0) = 1, \\
  g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\
  h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1.
\end{align*}
\]

Find an explicit formula for \( f(x) \), valid in some open interval around 0.

**Solution:**

Multiplying the first equation by \( gh \), the second equation by \( fh \), and the third equation by \( fg \), and summing gives us \( f'gh + fg'h + fgh' = 6(fgh)^2 + 6 \), or equivalently,

\[ (fgh)' = 6(fgh)^2 + 6. \]

This ordinary differential equation for \( fgh \) is solved by \( (fgh)(x) = \tan(6x + c) \) for some constant \( c \). The number \( c \) must be chosen to satisfy the initial condition \( (fgh)(0) = f(0)g(0)h(0) = 1 \), i.e., \( \tan(c) = 1 \). Hence we may take \( c = \frac{\pi}{4} \). By standard uniqueness, the following is valid in some open interval \( I \) around 0:

\[ f(x)g(x)h(x) = \tan(6x + \frac{\pi}{4}). \]

Since \( f \) is nonzero at 0, we may shrink \( I \) if necessary to assume that \( f \) does not vanish on \( I \). Then we can divide the first given equation by \( f \) to get

\[
\frac{f'(x)}{f(x)} = 2(fgh)(x) + \frac{1}{(fgh)(x)} = 2\tan(6x + \frac{\pi}{4}) + \cot(6x + \frac{\pi}{4}).
\]
Integrating both sides gives
\[
\ln (f(x)) = -\frac{1}{3} \ln (\cos(6x + \frac{\pi}{4})) + \frac{1}{6} \ln (\sin(6x + \frac{\pi}{4})) + d
\]
for some constant \(d\). This can be rewritten as
\[
f(x) = e^d \left( \frac{\sin(6x + \frac{\pi}{4})}{\cos^2(6x + \frac{\pi}{4})} \right)^{1/6}.
\]
The initial condition \(f(0) = 1\) implies that \(e^d \cdot 2^{1/12} = 1\), i.e., \(e^d = 2^{-1/12}\).

**Problem 3.** (2011 B3) Let \(f\) and \(g\) be (real-valued) functions defined on an open interval containing 0, with \(g\) nonzero and continuous at 0. If \(fg\) and \(f/g\) are differentiable at 0, must \(f\) be differentiable at 0?

**Solution 1:**
On an open interval containing 0, we have the following identity:
\[
\frac{f(h) - f(0)}{h} = \frac{f(h)^2 - f(0)^2}{h} \cdot \frac{1}{f(h) + f(0)}.
\]
(4)
Since \(fg\) and \(f/g\) are differentiable at 0, as is \((fg) \cdot (f/g) = f^2\). By definition, this means that \(\lim_{h \to 0} \frac{f(h)^2 - f(0)^2}{h}\) exists. On the other hand, \(f = (f/g) \cdot g\) is continuous at 0, since both \(f/g\) and \(g\) are continuous at 0. If \(f(0) \neq 0\), this implies that \(\lim_{h \to 0} \frac{1}{f(h) + f(0)} = \frac{1}{2f(0)}\). Hence if \(f(0) \neq 0\), (4) implies existence of \(\lim_{h \to 0} \frac{f(h) - f(0)}{h}\), which is equivalent to differentiability of \(f\) at 0.

If \(f(0) = 0\), \((f/g)(0) = 0\). Then differentiability of \(f/g\) at 0 implies that the following limit exists:
\[
\lim_{h \to 0} \frac{(f/g)(h) - (f/g)(0)}{h} = \lim_{h \to 0} \frac{f(h)}{hg(h)}.
\]
On the other hand, continuity of \(g\) at 0 implies that \(\lim_{h \to 0} g(h)\) exists. Multiplying these two limits, we obtain
\[
\left( \lim_{h \to 0} \frac{f(h)}{hg(h)} \right) \left( \lim_{h \to 0} g(h) \right) = \lim_{h \to 0} \left( \frac{f(h)}{hg(h)} \cdot g(h) \right) = \lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h},
\]
and thereby deduce differentiability of \(f\) at 0.

**Solution 2:**
We provide a different solution for the case when \(f(0) \neq 0\). Note that \(f\) is differentiable at 0 if and only if \(-f\) is differentiable at 0. Hence we may only consider the case when \(f(0) > 0\). Now \(f = (f/g) \cdot g\) is continuous at 0 as a product of two functions which are both continuous at 0, so \(f\) is always positive in some neighborhood of 0. Then we may write \(f = \sqrt{(fg) \cdot (f/g)}\) in this neighborhood. The square root function is
differentiable on \((0, \infty)\) while the functions \(fg\) and \(f/g\) are differentiable at 0. Hence the chain rule implies that \(f\) is differentiable at 0.

**Problem 4.** (2005 B3) Find all differentiable functions \(f : (0, \infty) \to (0, \infty)\) for which there is a positive real number \(a\) such that

\[
f'(a x) = \frac{x}{f(x)} \quad \text{for all } x > 0.
\]

**Solution:**

Let us first consider the functions of the form \(f(x) = cx^d\) where \(c, d > 0\). Then

\[
f'(\frac{a}{x}) = \frac{cda^{d-1}}{x^{d-1}}, \quad \frac{x}{f(x)} = \frac{1}{cx^{d-1}}.
\]

We want to choose \(a > 0\) so that these two are always equal, which means \(cda^{d-1} = \frac{1}{cx^{d-1}}\), or equivalently \(c^2d = 1\). If \(d \neq 1\), we can solve for \(a\); if \(d = 1\), we must have \(c = 1\).

We claim that these are all solutions. This means that \(\ln(f(x)) = d \ln x + c\) is a linear function of \(\ln x\). Setting \(t = \ln x\), we can rewrite this as \(\ln(f(e^t)) = dt + c\). Motivated by this, we define \(g : \mathbb{R} \to (0, \infty)\) by

\[
g(t) = \ln(f(e^t)).
\]  

We wish to show that \(g\) is a linear function. For this, set \(b = \ln a\) and write the given equation as

\[
f'(e^{b-t}) = \frac{e^t}{f(e^t)}. \quad (6)
\]

Since \(f\) is differentiable, applying the chain rule to (5) gives \(g'(t) = \frac{f'(e^t)e^t}{f(e^t)}\), or equivalently

\[
f'(e^t) = e^{-t}g'(t)f(e^t).
\]

This identity holds for all \(t \in \mathbb{R}\), so we have \(f'(e^{b-t}) = e^{t-b}g'(b-t)f(e^{b-t})\). Substituting this into (6) yields

\[
e^{t-b}g'(b-t)f(e^{b-t}) = \frac{e^t}{f(e^t)}.
\]

By definition of \(g\) in (5), taking the log on both sides gives

\[
t - b + \ln g'(b-t) + g(b-t) = t - g(t),
\]

or

\[
\ln g'(b-t) = b - (g(t) + g(b-t)). \quad (7)
\]
By symmetry of the right side, we observe that $g'(b - t) = g'(t)$. Then the derivative of the function $g(t) + g(b - t)$ is $g'(t) - g'(b - t) = 0$, implying that $g(t) + g(b - t)$ is constant. Hence it follows from (7) that $g'$ is also constant, which establishes the claim that $g$ is a linear function.