HOMEWORK 4 SOLUTIONS

Problem 1. (1999 A2) Let \( p(x) \) be a polynomial that is non-negative for all \( x \). Prove that, for some \( k \), there are polynomials \( f_1(x), \ldots, f_k(x) \) such that

\[
p(x) = \sum_{j=1}^{k} (f_j(x))^2.
\]

Solution 1:

By the Fundamental Theorem of Algebra, we may write

\[
p(x) = \prod_{i=1}^{k} (x - r_i)^{m_i} \prod_{j=1}^{l} \left( (x - s_j)^{n_j} (x - \bar{s}_j)^{n_j} \right),
\]

(1)

where \( r_i \)'s are real roots with multiplicity \( m_i \), and \( s_j \)'s are nonreal roots with multiplicity \( n_j \). Notice that for \( j = 1, 2, \ldots, l \), the polynomial \( (x - s_j)(x - \bar{s}_j) = x^2 - (s_j + \bar{s}_j)x + s_j \bar{s}_j \) is a quadratic polynomial with real coefficients. Hence by means of completing square we may write \( (x - s_j)(x - \bar{s}_j) = (x - a_j)^2 + c_j \) for some real numbers \( a_j \) and \( b_j \). Since the equation \( (x - a_j)^2 + b_j = 0 \) has two nonreal roots, we deduce that \( c_j > 0 \). Letting \( b_j = \sqrt{c_j} \), we rewrite (1) as

\[
p(x) = \prod_{i=1}^{k} (x - r_i)^{m_i} \prod_{j=1}^{l} \left( (x - a_j)^2 + b_j^2 \right)^{n_j}.
\]

(2)

Observe that each \( m_i \) must be an even number, otherwise \( p(x) \) would have a sign change at the corresponding root. This means that we can write \( \prod_{i=1}^{k} (x - r_i)^{m_i} = f(x)^2 \), where \( f(x) = \prod_{i=1}^{k} (x - r_i)^{m_i/2} \). Substituting this into (2), we deduce

\[
p(x) = f(x)^2 \prod_{j=1}^{l} \left( (x - a_j)^2 + b_j^2 \right)^{n_j}.
\]

By fully expanding the right hand side, we obtain a desired expression of \( p(x) \) as a sum of squares.

Solution 2:

As in Solution 1, consider the factorization (1). Note that we can write \( \prod_{j=1}^{l} (x - s_j)^{n_j} = g_1(x) + ig_2(x) \) for some polynomials \( g_1(x) \) and \( g_2(x) \) with real coefficients; in fact, \( g_1(x) \) and \( g_2(x) \) are respectively the real and the imaginary part of the polynomial \( \prod_{j=1}^{l} (x - s_j)^{n_j} \). Then we have

\[
\prod_{j=1}^{l} (x - s_j)^{n_j} = \prod_{j=1}^{l} (x - s_j)^{n_j} = g_1(x) + ig_2(x) = g_1(x) - ig_2(x).
\]
Substituting into (1), we obtain
\[ p(x) = \prod_{i=1}^{k} (x - r_i)^{m_i} \left( g_1(x) + ig_2(x) \right) \left( g_1(x) - ig_2(x) \right) = \left( \prod_{i=1}^{k} (x - r_i)^{m_i} \right) (g_1(x)^2 + g_2(x)^2). \tag{3} \]

Now as in Solution 1, we may write \( \prod_{i=1}^{k} (x - r_i)^{m_i} = f(x)^2 \) where \( f(x) = \prod_{i=1}^{k} (x - r_i)^{m_i/2} \), by observing that \( m_i \)’s are all even. Substituting this into (3), we obtain
\[ p(x) = f(x)^2 (g_1(x) + g_2(x))^2 = (f(x)g_1(x))^2 + (f(x)g_2(x))^2. \]

**Problem 2.** (2001 A3) For each integer \( m \), consider the polynomial
\[ P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2. \]

For what values of \( m \) is \( P_m(x) \) the product of two non-constant polynomials with integer coefficients?

**Solution 1:**

Since \( P_m(x) = (x^2 - (m + 2))^2 - 8m \), the roots of \( P_m(x) \) are given by \( \pm \sqrt{m + 2} \pm 2\sqrt{2m} = \pm \sqrt{m} \pm \sqrt{2}. \)

Suppose that \( P_m(x) = Q(x)R(x) \) where \( Q(x) \) and \( R(x) \) are nonconstant polynomials of integer coefficients. Since the leading coefficient for \( P_m(x) \) is 1, the leading coefficients for \( Q(x) \) and \( R(x) \) are either both 1 or both -1. Multiplying both factors by -1 if necessary, we may assume that both \( Q(x) \) and \( R(x) \) have a leading coefficient 1.

Consider the case when either \( Q(x) \) or \( R(x) \) has degree 1. Without loss of generality, we may assume that \( Q(x) \) has degree 1. Then \( Q(x) = x - r \) for some root \( r \) of \( P_m(x) \), so \( r \) must be an integer. Note that \( (\pm \sqrt{m} \pm \sqrt{2})^2 = m + 2 \pm 2\sqrt{2m} \), so \( 2m \) must be a square number, which implies that \( m = 2n^2 \) for some integer \( n \). Then \( \pm \sqrt{m} \pm \sqrt{2} = (n \pm 1)\sqrt{2} \), which is integer only if \( n = \pm 1 \), i.e., \( m = 2 \). When \( m = 2 \), we indeed have a desired factorization as \( P_m(x) = x^4 - 8x^2 = x^2(x^2 - 8) \).

Now consider the case when both \( Q(x) \) and \( R(x) \) are quadratic polynomials. Without loss of generality, we may assume that \( \sqrt{m} + \sqrt{2} \) is a root of \( Q(x) \). Let \( s \) be the other root of \( Q(x) \), then
\[ Q(x) = (x - \sqrt{m} - \sqrt{2})(x - s) = x^2 - (\sqrt{m} + \sqrt{2} + s)x + s(\sqrt{m} + \sqrt{2}). \]

Note that \( s \neq -\sqrt{m} + \sqrt{2} \), otherwise the coefficient of \( x \) in is not an integer as \( \sqrt{m} + \sqrt{2} + (-\sqrt{m} + \sqrt{2}) = 2\sqrt{2} \). This means that \( -\sqrt{m} + \sqrt{2} \) is a root of \( R(x) \). Let \( t \) be the other root of \( R(x) \), then
\[ R(x) = (x - (-\sqrt{m} + \sqrt{2}))(x - t) = x^2 - (-\sqrt{m} + \sqrt{2} + t)x + t(-\sqrt{m} + \sqrt{2}). \]
We have two possibilities: $s = \sqrt{m} - \sqrt{2}$, $t = -\sqrt{m} - \sqrt{2}$ or $s = -\sqrt{m} - \sqrt{2}$, $t = \sqrt{m} - \sqrt{2}$. For the first case, we have $Q(x) = x^2 - 2\sqrt{m}x + (m - 2)$ and $R(x) = x^2 + 2\sqrt{m}x + (m - 2)$, both of which have integer coefficients if and only if $2\sqrt{m}$ is an integer, i.e., $m$ is a perfect square. For the second case, we have $Q(x) = x^2 + m + 2 + 2\sqrt{2m}$ and $R(x) = x^2 - (m + 2 - 2\sqrt{2m})$, both of which have integer coefficients if and only if $2\sqrt{2m}$ is an integer, i.e., $m$ is twice a perfect square. Hence we conclude that $m$ is either a perfect square or twice a perfect square.

In sum, $P_m(x)$ is the product of two non-constant polynomials with integer coefficients if and only if $m$ is either a perfect square or twice a perfect square.

**Solution 2:**

As in Solution 1, we write $P(x) = Q(x)R(x)$ where $Q(x)$ and $R(x)$ are both monic polynomials with integer coefficients. We present a different solution for the case when both $Q(x)$ and $R(x)$ are quadratic polynomials.

Write $Q(x) = x^2 + rx + s$ and $R(x) = x^2 + r'x + t$ for integers $r, r', s, t$. Compairing the coefficient of $x^3$ in $P_m(x) = Q(x)R(x)$, we immediately see that $r + r' = 0$, i.e., $r' = -r$. Then

$$Q(x)R(x) = (x^2 + rx + s)(x^2 - rx + t) = x^4 + (s + t - r^2)x^2 + r(t - s)x + st.$$ 

Comparing this with $P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2$ yields three equations:

$$r^2 - s - t = 2m + 4, \quad r(t - s) = 0, \quad st = (m - 2)^2. \quad (4)$$

Hence existence of the factorization of $P_m(x)$ into monic quadratic polynomials with integer coefficients is equivalent to existence of integers $r, s, t$ satisfying the above equations.

The second equation $r(t - s) = 0$ implies that there are two possibilities: $r = 0$ or $s = t$.

When $r = 0$, the equations in (4) reduce to $s + t = -(2m + 4)$ and $st = (m - 2)^2$. Then $s$ and $t$ are roots of the quadratic polynomial $x^2 + (2m + 4)x + (m - 2)^2$. By the quadratic formula, the roots are given by $-(m + 2) \pm \sqrt{(m + 2)^2 - (m - 2)^2} = -(m + 2) \pm 2\sqrt{2m}$. Thus there is an integer solution to (4) if and only if $\sqrt{2m}$ is an integer, i.e., $m$ is twice a perfect square.

For the case when $s = t$, the equations in (4) reduce to $r^2 - 2s = 2m + 4$ and $s^2 = (m - 2)^2$. The latter gives $s = \pm(m - 2)$, and substituting this into the former yields $r^2 = 8$ or $r^2 = 4m$. Thus there is an integer solution to (4) if and only if $m$ is a perfect square.
Problem 3. (2003 B1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Solution 1:

For $i = 0, 1, 2$, let $a_i$ be the coefficient of $x^i$ in $a(x)$. Similarly we define $b_i, c_i, d_i$ for $i = 0, 1, 2$. For any $i, j \in \{0, 1, 2\}$, comparing the coefficients of $x^i y^j$ in $1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$ yields

$$a_i c_j + b_i d_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Now consider the two vectors $v = (c_0, c_1, c_2)$ and $w = (d_0, d_1, d_2)$ in $\mathbb{R}^3$. Let $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, $e_2 = (0, 0, 1)$ denote the standard basis vectors. The equations in (5) can be written as

$$a_i v + b_i w = c_i \quad \text{for } i = 0, 1, 2.$$ 

For these equations to have a solution, the vectors $v$ and $w$ must span the three dimensional space $\mathbb{R}^3$ as they span the standard basis vectors. However, this is impossible since two vectors can never span a three dimensional space. Thus there do not exist such polynomials $a(x), b(x), c(y), d(y)$.

Solution 2:

We give a different proof that there is no solutions to the equations in (5).

For $i = 0, 1, 2$, the equation $a_i c_i + b_i d_i = 1$ says that $a_i$ and $b_i$ cannot both vanish, and $c_i$ and $d_i$ cannot both vanish. Without loss of generality, we may assume that $b_0 \neq 0$. From $a_0 c_1 + b_0 d_1 = 0$, we get $d_1 = -\frac{a_0}{b_0} c_1$.

Note that $c_1 \neq 0$ otherwise $c_1$ and $d_1$ would both vanish. Now substituting this into $a_2 c_1 + b_2 d_1 = 0$ yields $a_2 c_1 + b_2 \frac{a_0}{b_0} c_1 = 0$. As $c_1 \neq 0$, we obtain $a_2 = -\frac{a_0}{b_0} b_2$. Again, we note that $b_2 \neq 0$ otherwise $a_2$ and $b_2$ would both vanish. Then we substitute this into $a_2 c_0 + b_2 d_0 = 0$ to get $-\frac{a_0}{b_0} b_2 c_0 + b_2 d_0 = 0$. Multiplying both sides by $\frac{b_0}{b_2}$, which is possible since $b_2 \neq 0$, we obtain $a_0 c_0 + b_0 d_0 = 0$, which is a contradiction. Hence we conclude that there is no solutions to the equations in (5).

Remark. Solution 2 can be (informally) written as follows. From (5), we have $a_i b_j = -\frac{c_i}{d_j}$ if $i \neq j$, where we allow 0 and $\infty$ in both sides. Then we have

$$\frac{a_0}{b_0} = -\frac{c_1}{d_1} = \frac{a_2}{b_2} = -\frac{c_0}{d_0},$$

which yields $a_0 c_0 + b_0 d_0 = 0$. 

Problem 4. (2007 A4) A repunit is a positive integer whose digits in base 10 are all ones. Find all polynomials \( f \) with real coefficients such that if \( n \) is a repunit, then so is \( f(n) \).

Solution:

A repunit is a number of the form \( \frac{10^n - 1}{9} \) for some positive integer \( m \). Hence for every positive integer \( r \), there exists a positive integer \( s \) such that \( f \left( \frac{10^r - 1}{9} \right) = \frac{10^s - 1}{9} \), or equivalently \( 9f \left( \frac{10^r - 1}{9} \right) + 1 = 10^s \).

Take \( g(x) = 9f \left( \frac{x - 1}{9} \right) + 1 \), which is a polynomial with real coefficients. Then the above identity can be written as \( g(10^r) = 10^s \). This implies that \( f \) takes repunits to repunits if and only if \( g \) takes powers of 10 greater than 1 to powers of 10 greater than 1.

We claim that \( g(x) = 10^c x^d \) for some nonnegative integers \( c, d \), at least one of which is nonzero. Let \( ax^d \) be the leading term of \( g(x) \). Notice that \( a > 0 \), otherwise there exists \( N > 0 \) such that \( g(x) \leq 0 \) for all \( x > N \), which is impossible since \( g(10^r) > 0 \) for all positive integers \( r \). We further note that \( \lim_{x \to \infty} \frac{g(x)}{x^d} = a \), which gives

\[
\lim_{r \to \infty} \frac{g(10^r)}{10^{rd}} = a. \tag{6}
\]

If \( a \) is not a power of 10, there exists a unique positive integer \( b \) such that \( 10^{b-1} < a < 10^b \). Then the limit (6) says that there exists \( M > 0 \) such that for all integers \( r > M \) we have \( 10^{b-1} < \frac{g(10^r)}{10^{rd}} < 10^b \), i.e., \( 10^{rd+b-1} < g(10^r) < 10^{rd+b} \). This is, however, impossible since there is no power of 10 between \( 10^{rd+b-1} \) and \( 10^{rd+b} \). Hence we conclude by contradiction that \( a \) is a power of 10. Now we may write \( a = 10^c \) with some nonnegative integer \( c \). Since \( 10^{c-1} < a < 10^{c+1} \), the limit (6) gives a number \( M' \) such that for all integers \( r > M' \) we have \( 10^{c-1} < \frac{g(10^r)}{10^{rd}} < 10^{c+1} \), i.e., \( 10^{rd+c-1} < g(10^r) < 10^{rd+c+1} \). Then \( g(10^r) \) is a power of 10 between \( 10^{rd+c-1} \) and \( 10^{rd+c+1} \), so we must have \( g(10^r) = 10^{rd+c} \) for all integers \( r > M' \). This means that the equation \( g(x) = 10^c x^d \) has infinitely many solutions, namely \( x = 10^r \) for all \( r > M' \). Hence we must have \( g(x) = 10^c x^d \), as desired. Note that \( c \) and \( d \) cannot both vanish, otherwise \( g(10^r) = 1 \) for all integers \( r \).

It is evident that all polynomials of this form satisfy the condition specified in the first paragraph. Now we can recover \( f(x) \) from \( g(x) \) by

\[
f(x) = \frac{1}{9} (g(9x + 1) - 1) = \frac{1}{9} (10^c (9x + 1)^d - 1).
\]

where \( c \) and \( d \) are nonnegative integers, at least one of which is nonzero.