Problem 1. (2002 B2) Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

Solution 1:

Let $v, e, f$ be as in Euler’s formula $v - e + f = 2$. In other words, $v, e$ and $f$ are respectively the number of vertices, edges and faces.

For each vertex, we define its degree to be the number of edges emerging from it. Then the sum of the degrees of all vertices is $3v$, since each vertex is of degree 3. On the other hand, the sum of the degrees of all vertices is equal to $2e$, as each edge is counted twice in this sum. Hence we have $3v = 2e$, or equivalently $e = \frac{3}{2}v$. Substituting this into Euler’s formula yields $-\frac{v}{2} + f = 2$, which we can rewrite as $v = 2f - 4$.

For each face, we define its degree to be the number of edges bordering it. Note that this is the same as the number of vertices the face has on its border. Each vertex has degree 3, so each vertex is shared by three faces. Hence the sum of the degrees of all faces is $3v$, which is also equal to $3(2f - 4) = 6f - 12$. Note that this is larger than $3f$ as $f > 4$. Hence we deduce that there exists a face of degree larger than 3.

Let $A$ be a face of degree larger than 4. We claim that the first player will always win by signing on $A$ first. Indeed, after the second player’s first move there will be three consecutive faces $B, C, D$ adjacent to $A$ which are all unsigned. Then the first player ensures his win by signing on $C$.

Solution 2:

We give a different proof that the polyhedron has a face with at least four edges. Suppose for contradiction that each face has only three edges. Let $F$ be a face with vertices $v_1, v_2$ and $v_3$. Let $F_1, F_2, F_3$ be the faces adjacent to $F$ such that $F_i$ and $F$ shares $v_i$ and $v_{i+1}$ for $i = 1, 2, 3$, where we define $v_4 = v_1$ for convenience. For $i = 1, 2, 3$, let $w_i$ be the vertex of $F_i$ not on $F$. Note that $v_1$ is adjacent to $v_2, v_3, w_1$ and $w_3$. Since there are only three edges emerging from $v_1$, we must have $w_1 = w_3$. Similarly, by looking at vertices adjacent to $v_2$, we deduce $w_1 = w_2$. Hence we have $w_1 = w_2 = w_3$. Thus the four faces $F, F_1, F_2, F_3$ form a polyhedron by themselves, contradicting the fact that the given polyhedron has at least five faces.
Problem 2. (2007 A2) Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola \( xy = 1 \) and both branches of the hyperbola \( xy = -1 \). (A set \( S \) in the plane is called convex if for any two points in \( S \) the line segment connecting them is contained in \( S \).)

Solution 1:

Let \( S \) be a convex set with the desired properties. Let \( A \) be a point in \( S \) that lies on the upper right branch of the hyperbola \( xy = 1 \), Similarly, we choose points \( B \in S \) on the upper left quadrant, \( C \in S \) on the lower left quadrant, and \( D \in S \) on the lower right quadrant. \( S \) contains the quadrilateral \( ABCD \) due to convexity, so \( \text{Area}(S) \geq \text{Area}(\square ABCD) \).

Write \( A = (a, \frac{1}{a}), B = (-b, \frac{1}{b}), C = (-c, -\frac{1}{c}), D = (d, -\frac{1}{d}) \) with \( a, b, c, d > 0 \). Let \( \theta \) be the angle between two line segments \( AC \) and \( BD \), then

\[
\text{Area}(\square ABCD) = \frac{1}{2} |AC| \cdot |BD| \sin \theta. \tag{1}
\]

On the other hand, we have

\[
|AC| = \sqrt{(a+c)^2 + \left(\frac{1}{a} + \frac{1}{c}\right)^2} = \sqrt{a^2 + c^2 + ac + \frac{1}{a^2} + \frac{1}{c^2} + \frac{1}{ac}},
\]

\[
\geq \sqrt{8 \left(\frac{a^2 \cdot c^2 \cdot ac \cdot \frac{1}{a^2} \cdot \frac{1}{c^2} \cdot \frac{1}{ac}}{a^2 + c^2 + ac}\right)^{1/8}} = \sqrt[8]{8} = 2\sqrt{2},
\]

where the inequality follows from AM-GM. The equality holds when \( a = c = 1 \). Similarly, we get \( |BD| \geq 2\sqrt{2} \), with the equality when \( b = d = 1 \). Since \( 0 \leq \sin \theta \leq 1 \), (1) yields

\[
\text{Area}(\square ABCD) \geq \frac{1}{2} \cdot 2\sqrt{2} \cdot 2\sqrt{2} = 4.
\]

It is evident that the equality holds when \( a = b = c = d = 1 \). Hence we conclude \( \text{Area}(S) \geq 4 \), where the minimum is attained when \( S \) is the quadrilateral with vertices \((1, 1), (-1, 1), (-1, -1)\) and \((1, -1)\).

Solution 2:

Let \( A, B, C, D \) as in Solution 1. We present another proof of the inequality \( \text{Area}(\square ABCD) \geq 4 \).

By working on the complex plane, we may represent \( A, B, C, D \) by four complex numbers \( z_A, z_B, z_C, z_D \) respectively. Write \( z_A = a + i\frac{1}{a}, z_B = -b + i\frac{1}{b}, z_C = -c - i\frac{1}{c}, z_D = d - i\frac{1}{d} \) with \( a, b, c, d > 0 \). Then
\[ z_A - z_C = (a + c) + i \left( \frac{1}{a} + \frac{1}{c} \right) \text{ and } z_B - z_D = -(b + d) + i \left( \frac{1}{b} + \frac{1}{c} \right). \] Hence we have

\[
\text{Area}(ABCD) = \frac{1}{2} \left| \det \begin{pmatrix} a + c & \frac{1}{a} + \frac{1}{c} \\ -(b + d) & \frac{1}{b} + \frac{1}{d} \end{pmatrix} \right|
\]

\[
= \frac{1}{2} \left( (a + c) \left( \frac{1}{b} + \frac{1}{d} \right) + (b + d) \left( \frac{1}{a} + \frac{1}{c} \right) \right)
\]

\[
= \frac{1}{2} \left( \frac{a}{b} + c + \frac{a}{d} + c + \frac{b}{a} + \frac{b}{c} + \frac{d}{a} + \frac{d}{c} \right)
\]

\[
\geq \frac{1}{2} \cdot 8 \left( \frac{a}{b} \cdot \frac{c}{b} \cdot \frac{c}{d} \cdot \frac{b}{a} \cdot \frac{d}{a} \cdot \frac{d}{c} \right)^{1/8} = 4
\]

where the inequality follows from AM-GM. The equality holds when \( a = b = c = d = 1. \)

**Problem 3.** (2004 B4) Let \( n \) be a positive integer, \( n \geq 2 \), and put \( \theta = 2\pi/n \). Define points \( P_k = (k, 0) \) in the \( xy \)-plane, for \( k = 1, 2, \cdots, n \). Let \( R_k \) be the map that rotates the plane counterclockwise by the angle \( \theta \) about the point \( P_k \). Let \( R \) denote the map obtained by applying, in order, \( R_1 \), then \( R_2, \cdots, \), then \( R_n \). For an arbitrary point \((x, y)\), find, and simplify, the coordinates of \( R(x, y) \).

**Solution:**

We will work on the complex plane. Let \( \zeta = e^{i\theta} \). For \( k = 1, 2, \cdots, n \), the point \( P_k \) corresponds to the real number \( k \). We can also describe the map \( R_k \) by

\[
R_k(z) = \zeta(z - k) + k \text{ for all } z \in \mathbb{C}. \tag{2}
\]

Now we claim that \( R_k \circ R_{k-1} \circ \cdots \circ R_1(z) = \zeta^k z - (\zeta + \zeta^{-1} + \cdots + \zeta) + k \) for \( k = 1, 2, \cdots, n \). We prove this by induction on \( k \). The case when \( k = 1 \) is trivial by description of \( R_1 \) given in (2). For inductive step, we suppose that the above equation is true for \( k = i \). Then we calculate

\[
R_{i+1} \circ R_i \circ \cdots \circ R_1(z) = R_{i+1}(\zeta^i z - (\zeta^i + \zeta^{-1} + \cdots + \zeta) + i)
\]

\[
= \zeta(\zeta^i z - (\zeta^i + \zeta^{-1} + \cdots + \zeta) + i - (i + 1)) + (i + 1)
\]

\[
= \zeta(\zeta^i z - (\zeta^i + \zeta^{-1} + \cdots + \zeta + 1)) + (i + 1)
\]

\[
= \zeta^{i+1} z - (\zeta^{i+1} + \zeta^i + \cdots + \zeta) + (i + 1),
\]

where the second identity follows from (2) with \( k = i + 1 \). This completes the inductive step.
As a special case of the above claim, we have

\[ R(z) = R_n \circ R_{n-1} \circ \cdots \circ R_1(z) \]
\[ = \zeta^n z - (\zeta^n + \zeta^{n-1} + \cdots + \zeta) + n \]
\[ = \zeta^n z - \zeta(\zeta^{n-1} + \zeta^{n-2} + \cdots + 1) + n \]
\[ = z + n, \]

where the last equality follows from the well-known identities such as \( \zeta^n = 1 \) and \( \zeta^{n-1} + \zeta^{n-2} + \cdots + 1 = 0 \). In terms of \( xy \)-coordinates, \( R \) maps \((x, y)\) to \((x + n, y)\).

**Remark.** The identities \( \zeta^n = 1 \) and \( \zeta^{n-1} + \zeta^{n-2} + \cdots + 1 = 0 \) can be seen as follows. First, we have \( \zeta^n = (e^{i\theta})^n = e^{in\theta} = e^{i2\pi/n} = 1 \) as \( \theta = 2\pi/n \). From this we obtain \((\zeta - 1)(\zeta^{n-1} + \zeta^{n-2} + \cdots + \zeta + 1) = \zeta^n - 1 = 0\), which further yields \( \zeta^{n-1} + \zeta^{n-2} + \cdots + 1 = 0 \) since \( \zeta \neq 1 \).

**Problem 4.** (2004 A2) For \( i = 1, 2 \) let \( T_i \) be a triangle with side lengths \( a_i, b_i, c_i \), and area \( A_i \). Suppose that \( a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2 \), and that \( T_2 \) is an acute triangle. Does it follow that \( A_1 \leq A_2 \)?

**Solution:**

We will prove that \( A_1 \leq A_2 \).

First, we reduce to the case when \( a_1 = a_2 \). Consider the numbers \( \frac{a_2}{a_1}, \frac{b_2}{b_1}, \frac{c_2}{c_1} \). After relabeling if necessary, we may assume that \( \frac{a_2}{a_1} \) is the smallest among these numbers. Let \( T'_1 \) be a triangle with side lengths \( \frac{a_2}{a_1}, \frac{a_2}{a_1}, \frac{a_2}{a_1} \), and area \( A'_1 \). By the assumption that \( \frac{a_2}{a_1} \) is the smallest among the numbers \( \frac{a_2}{a_1}, \frac{b_2}{b_1}, \frac{c_2}{c_1} \),

\[ \frac{a_2}{a_1} a_2 b_1 \leq b_2, \frac{a_2}{a_1} c_1 \leq c_2. \]  

(3)

On the other hand, since \( T'_1 \) and \( T_1 \) are similar triangles with similarity ratio \( \frac{a_2}{a_1} \), we have \( A'_1 = \left( \frac{a_2}{a_1} \right)^2 A_1 \). This yields \( A'_1 \geq A_1 \) as \( a_2 \geq a_1 \), so it suffices to prove that \( A'_1 \leq A_2 \). By (3), we have reduced to the case when \( a_1 = a_2 \).

Now we consider the case when \( a_1 = a_2 \). For the sake of contradiction, we assume that \( A_1 > A_2 \) for some triangles \( T_1 \) and \( T_2 \) with \( a_1 = a_2, b_1 \leq b_2, c_1 \leq c_2 \). Choose four points \( A, B, C, A' \) on the plane such that \( |BC| = a_1 = a_2, |CA| = b_1, |AB| = c_1, |CA'| = b_2, |A'B| = c_2 \). In other words, \( A, B, C, A' \) are chosen such that \( \triangle ABC \equiv T_1 \) and \( \triangle A'BC \equiv T_2 \).
Let $H$ (resp. $H'$) be the foot of the perpendicular from $A$ (resp. $A'$) to the line $BC$. Observe that $H'$ lines on the line segment $BC$ since $\triangle A'BC \equiv T_2$ is an acute triangle. Hence $H$ lies on either the ray $H'B$ or the ray $H'C$. Without loss of generality, we assume that $H$ lies on the ray $H'B$. This implies that $|CH| \geq |CH'|$, as $H'$ must lie on the line segment $CH$. We also note that $|AH| > |AH'|$ by our assumption that $A_1 > A_2$. Then we have

$$b_1 = |AC| = \sqrt{|AH|^2 + |CH|^2} > \sqrt{|AH'|^2 + |CH'|^2} = |A'C| = b_2,$$

which contradicts the fact that $b_1 \leq b_2$. 