1. Since $H$ is a nonempty subset of $G$, we just need to show that $\forall a, b \in H$, $a^{-1} \in H$ and $ab \in H$ (associativity always follows from $H$ being a subset and identity always follows from closure and inverse).

Since $H$ is nonempty, $\exists a \in H$. Then, $aa^{-1} \in H \implies 1 \in H$.

Since $1, a \in H$, we have that $1a^{-1} \in H \implies a^{-1} \in H$ and we have shown that $H$ contains inverses.

Finally, $\forall a, b \in H$, we have shown that $a, b^{-1} \in H$ and thus $a(b^{-1})^{-1} = ab \in H$ and we have shown that $H$ is closed under the group operation.

Therefore, since $H$ is a nonempty subset of $G$ which is closed under inverses and the group operation, it is a subgroup of $G$.

\[\square\]

2. We know that the number of distinct orbits of a group is given by

\[
\# \text{ Orbits} = \frac{1}{|G|} \sum_{g \in G} |F(g)|
\]

where in this case we are given that there are 2 distinct orbits. Assume for sake of contradiction that $\forall g \in G$ that $g$ contains at least one cycle of length one in its cycle structure. This is equivalent to saying that $g$ fixes at least one element or $\forall g \in G, F(g) \geq 1$. Consider $1 \in G$. Since $1$ fixes all elements of $X$, $F(1) = |X|$. Thus,

\[
\# \text{ Orbits} = \frac{1}{|G|} \sum_{g \in G} |F(g)|
\]

\[
= \frac{1}{|G|} (|X| + \sum_{g \in G \setminus \{1\}} |F(g)|)
\]

\[
\geq \frac{|X| + |G| - 1}{|G|}
\]

\[
= \frac{|X| - 1}{|G|} + 1
\]

\[
> 2
\]
By the assumption that $|X| - 2 \geq |G| \geq 2$. This is a contradiction because we know that the number of orbits is exactly 2, therefore there must exist some element of $G$ that does not contain any cycles of length one in its cycle structure.

$\square$

3. We know from lecture that the number of distinct colorings is the number of distinct orbits of the group of allowed symmetries of the octahedron. As in the previous problem, this is given by

$$\frac{1}{|G|} \sum_{g \in G} |F(g)|$$

As in the problem statement, let $u$ be the original vertex at the top and $v$ be the original vertex at the bottom. Number the other 4 vertices 1, 2, 3, 4.

First, we compute the order of the group $G$ of valid rotational symmetries of the octahedron. There are 2 choices for locations of vertex $u$, either the top or the bottom. After $u$ is placed, $v$ is fixed. Now, there are 4 locations for the vertex 1. After vertex 1 is placed, the locations of 2, 3, and 4 are likewise fixed if we are not allowed reflections of the octahedron (as explained on the website, $u$ and $v$ are switched through a rotation, not a reflection). Thus, $|G| = 4 \times 2 = 8$.

These 8 symmetries are
1) Identity
2) $90^\circ$ rotation in the plane of 1, 2, 3, 4
3) $180^\circ$ rotation
4) $270^\circ$ rotation
5) Rotate $u$ down to $v$ (doesn’t matter how, as long as it’s consistent with the other symmetries)
6) Rotate $u$ down to $v$ then rotate $90^\circ$
7) Rotate $u$ down to $v$ then rotate $180^\circ$
8) Rotate $u$ down to $v$ then rotate $270^\circ$

These are clearly all distinct symmetries, and since there are 8 of them, we know for sure we have all possible rotational symmetries.

Now we compute $F(g)$ for each of these symmetries.
1) This fixes all possible colorings of the octahedron. There are 8 faces that can be each colored with 3 different colors, therefore this fixes all $3^8$ colorings.
2, 4) For these two rotations, the entire top half of the octahedron must be the same color and likewise for the bottom half. So each of these fixes $3^2$ colorings.
3, 5, 6, 7, 8) Each of these rotations rotate a pair of faces into each other (application of any of rotations twice results in the identity rotation).
Each of these pairs of faces must be the same color. Since there are 4 such pairs, the number of fixed colorings for each of these is $3^4$.

Then, the number of distinct colorings under allowed rotations of this octahedron is

$$\frac{1}{|G|} \sum_{g \in G} |F(g)| = \frac{3^8 + 2 \cdot 3^2 + 5 \cdot 3^4}{8} = 873$$

4. We define the sequence $b_n = \log_2 a_n$. So $b_0 = 2$, $b_1 = 4$, and for $n \geq 2$ $b_n$ satisfies the recurrence relation $b_n = \frac{2}{3}b_{n-1} + \frac{1}{3}b_{n-2}$. So

$$B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$

$$B(x) = b_0 + b_1 x + \left( \frac{2}{3}b_1 + \frac{1}{3}b_0 \right)x^2 + \left( \frac{2}{3}b_2 + \frac{1}{3}b_1 \right)x^3 + \ldots$$

$$B(x) = b_0 + b_1 x - \frac{2}{3}b_0 x + \frac{2}{3}x \left( b_0 + b_1 x + \ldots \right) + \frac{x^2}{3} \left( b_0 + b_1 x + \ldots \right)$$

$$B(x) = b_0 + \left( b_1 - \frac{2b_0}{3} \right)x + \frac{2x}{3}B(x) + \frac{x^2}{3}B(x)$$

$$(1 - \frac{2x}{3} - \frac{x^2}{3})B(x) = b_0 + \left( b_1 - \frac{2b_0}{3} \right)x$$

$$(3 - 2x - x^2)B(x) = 3b_0 + (3b_1 - 2b_0)x$$

$$B(x) = \frac{3b_0 + (3b_1 - 2b_0)x}{3 - 2x - x^2} = \frac{6 + 8x}{3 - 2x - x^2}$$

since our initial conditions were $b_0 = 2$ and $b_1 = 4$. Since we can factor $3-2x-x^2$ into $(1-x)(3+x)$, we can perform a partial fractions decomposition:

$$\frac{6 + 8x}{3 - 2x - x^2} = \frac{A}{1-x} + \frac{B}{3+x}$$

$$6 + 8x = A(3+x) + B(1-x)$$

If we plug in $x = 1$, we get that $A = \frac{7}{2}$, and if we plug in $x = -3$, we get $B = -\frac{9}{2}$. So

$$B(x) = \frac{\frac{7}{2}}{1-x} - \frac{\frac{9}{2}}{3+x} = \frac{\frac{7}{2}}{1-x} - \frac{\frac{3}{2}}{1+\frac{x}{3}}$$

We recognize the right hand side as a sum of sums of geometric sequences, so we can expand to get

$$B(x) = \frac{7}{2} \sum_{n=0}^{\infty} x^n - \frac{3}{2} \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^n x^n$$

$$B(x) = \sum_{n=0}^{\infty} \left( \frac{7}{2} - \frac{3}{2} \left( -\frac{1}{3} \right)^n \right) x^n$$

So we know that $b_n = \frac{7}{2} - \frac{3}{2} \left( -\frac{1}{3} \right)^n$. Since we defined $b_n = \log_2 a_n$, we must have $a_n = 2^{\frac{7}{2} - \frac{3}{2} \left( -\frac{1}{3} \right)^n}$. 3
5. **First Solution.** Consider the last bit of a proper sequence of length $n$. If it is a 0, then the sequence will be valid if and only if the first $n - 1$ are a proper sequence of length of $n - 1$; we have exactly $b_{n-1}$ such sequences. If the last bit is a 1, then if our sequence is to be proper, we must have had at least 3 consecutive 1s at the end of our sequence. If we had exactly $3$ 1s at the end, we know that our sequence ended in the four bits 0111. We can then fill the remaining $n - 4$ bits with any proper sequence of length $n - 4$, of which we have $b_{n-4}$. If we have strictly more than 3 1s at the end, our prefix string must have been a proper sequence of length $b_{n-1}$ that ended in a 1. If a sequence of length $n - 1$ ended in a 0, then we would have $b_{n-2}$ ways to fill in the remaining bits, so we have $b_{n-1} - b_{n-2}$ total sequences that work in this case. So our recursion is

$$b_n = 2b_{n-1} - b_{n-2} + b_{n-4}$$

We can easily calculate our initial conditions $b_0 = b_1 = b_2 = 0$ since we can only have 0s (or no bits at all), $b_3 = 2$ since 000 and 111 are the only valid sequences.

**Second Solution.** As above we consider the last bit of a proper sequence of length $n$. If the last bit is 0, then we can argue as above that there are exactly $b_{n-1}$ valid sequences in this case. If the last bit is a 1, then we must have at least the last 3 bits be 1. In this case, our prefix sequence could either have a valid sequence of length $n - 3$, of which we have exactly $b_{n-3}$, or a valid sequence of length $n - 5$ that has 01 appended to end, of which we have exactly $b_{n-5}$, or a valid sequence of length $n - 6$ that has 011 appended at the end, of which we have exactly $b_{n-6}$. We know we have covered all of the cases, since the first case covers the cases which our sequence of length $n$ has either exactly 3 1s at the end or at least 6 1s at the end; the second case covers the case where there are exactly 4 1s at the end, and the third case covers the case where there are exactly 5 1s at the end. So our recurrence relation is

$$b_n = b_{n-1} + b_{n-3} + b_{n-5} + b_{n-6}$$

Our initial conditions are $b_0 = b_1 = b_2 = 1$, $b_3 = 2$, $b_4 = 4$, and $b_5 = 7$.

6. Consider the first move we make. If our first move if $(0, 0) \rightarrow (1, 1)$, then we must get from $(1, 1)$ to $(n, n)$ without exiting the triangular integer lattice $\{(x, y) \in \mathbb{N} | 1 \leq x \leq y \leq n\}$, of which there are exactly $c_{n-1}$ ways to do so. If our first move is $(0, 0) \rightarrow (1, 0)$, then let $(i, i)$ be the first point at which we hit the diagonal. We must hit the diagonal eventually, since our path must terminate at $(n, n)$. Since no point $(i', i')$ with $i' < i$ is on our path, we know we must have reached $(i, i)$ from below. So our the number of ways to reach $(i, i)$ without touching the diagonal at any intermediate is the same as the number of paths from $(0, 0)$ to $(i-1, i-1)$ that do not exit the lattice, which is $c_{i-1}$. The number of paths from $(i, i)$ to $(n, n)$ that do not exit the lattice is the same as the number of paths
from \((0,0)\) to \((n-i, n-i)\) that do not exit the lattice, so we have \(c_{n-i}\) paths. Thus we have \(c_{i-1}c_{n-i}\). Since \(i\) ranges over 1 to \(n\), our recurrence relation is

\[
c_n = c_{n-1} + \sum_{k=1}^{n} c_{i-1}c_{n-i}
\]

with the initial condition that \(c_0 = 0\), since there is exactly one way to get from \((0, 0)\) to \((0, 0)\) without exiting the point \((0, 0)\) (which is the entirety of our lattice in this case): stay in place (i.e. our sequence of moves is the empty sequence).