1. For integers $0 < r < n$, express $\sum_{k=r}^{n} \binom{k}{r}$ as a single binomial coefficient. Explain your answer.

**Method 1 (Induction):** We will show that $\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$. Take the base case where $n = r$.

$$\sum_{k=r}^{r} \binom{k}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1}.$$  

Now consider the for the inductive step that $\sum_{k=r}^{m} \binom{k}{r} = \binom{m+1}{r+1}$ for some $m$. Then

$$\sum_{k=r}^{m+1} \binom{k}{r} = \sum_{k=r}^{m} \binom{k}{r} + \binom{m+1}{r} = \binom{m+1}{r+1} + \binom{m+1}{r} = \binom{m+2}{r+1},$$

by the Pascal Identity. This concludes our inductive step, concluding the proof.

**Method 2 (Combinatorial Argument):** Consider a set of numbers $S = \{1, 2, \ldots, n+1\}$. Then if we pick any set of $r+1$ elements from $S$, one of the elements must be between $\{1, 2, \ldots, n-r\}$. Assuming we take 1 as the lowest element, then there are $\binom{n}{r}$ ways of choosing the remaining $r$ elements. Next we assume we take 2 as the lowest element (hence we cannot take 1), then there are $\binom{n-1}{r}$ ways of choosing the remaining $r-1$ elements. We continue this process until we choose $n-r$ as the lowest element, then there is only $\binom{r}{r} = 1$ way of picking the remaining $r$ elements in the set. Notice that the number of ways of choosing $r+1$ elements from $n+1$ elements is given by

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n-1}{r} + \cdots + \binom{r}{r} = \sum_{k=r}^{n} \binom{k}{r},$$

as desired.

**Method 3 (General Explanation):** This problem did not require any proof, but there should be some explanation of how it is done. Explicitly showing that you can expand $\sum_{k=r}^{n} \binom{k}{r}$ and repeatedly using Pascal’s identity was sufficient.
2. Basic counting:
(a) For a positive integer $n$ satisfying $n \equiv 1 \mod 4$, how many subsets $S \subseteq \{1, 2, 3, \ldots, n\}$ satisfy that the sum of the numbers in $S$ is larger than the sum of the numbers not in $S$? Explain your answer.
(b) For positive integers $n$ and $m$, consider the integer lattice $\{(x, y) \in \mathbb{N} : 0 \leq x \leq n \text{ and } 0 \leq y \leq m\}$. We travel the lattice from the point $(0, 0)$ to the point $(n, m)$, such that at each step we either go one step to the right or one step up (that is, we either increase the $x$-coordinate by one, or we increase the $y$-coordinate by one). How many different paths are available to us? Explain your answer.

Solution. (a)
Adding up all the elements of $S$, we get $\frac{n(n+1)}{2}$ which must be odd since $n \equiv 1 \mod 4$. Explicitly, let $n = 4m + 1$ for some $m \in \mathbb{Z}$. Then $\frac{(4m+1)(4m+2)}{2} = \frac{16m^2 + 12m + 2}{2} = 8m^2 + 6m + 1$, odd. Take some subset $A \subseteq S$ and consider the complement of the subset, namely $A^c$. Then the sum of elements in $A$ will be of opposite parity of that of the sum of elements in $A^c$. Therefore either the sum of elements in $A$ is greater than the sum of elements in $A^c$ or vice versa. Half the subsets of $S$ will have a greater sum of elements than the other half. There are $2^n$ subsets of $S$, the number of sets satisfying that the sum of numbers in $S$ is larger than the sum of numbers not if $S$ is

$$\frac{2^n}{2} = 2^{n-1}$$

Solution. (b)
There are $n + m$ total moves we have to make. If we determine the $n$ steps we wish to go right then the other $m$ up steps are fully determined. Thus, we just need to count how many ways we can arrange $n$ steps to the right from a total $n + m$ steps. Namely,

$$\binom{n + m}{n} = \binom{n + m}{m},$$

equivalent by symmetry.

3. Prove the formula $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ for all $n \geq 1$ (hint: Start with the formula for $(x + y)^n$ and set $y = 1$).

Proof. Consider $(x + y)^n$. By the binomial theorem, taking $y = 1$, $(x + 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Let us take the derivative of $(x + 1)^n$ with respect to $x$. Hence

$$\frac{d}{dx} (x + 1)^n = n(x + 1)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1} = \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Take $x = 1$. Then we get $n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$, as desired.

\[ \square \]
4. Consider a connected undirected graph $G = (V, E)$ and two vertices $s, t \in V$, such that the shortest path between $s$ and $t$ is of length larger than $|V|/2$. Prove that there exists a vertex $v \in V \setminus \{s, t\}$ such that after removing $v$ from $G$ (together with the adjacent edges) there are no paths between $s$ and $t$ (hint: This question is related to the BFS algorithm).

**Solution.**
Consider a tree $T$ created by the BFS algorithm with $s$ as the root node. Then there are at least $\frac{|V|}{2} + 1$ levels between $s$ and $t$. Without loss of generality, let the tree have $\frac{|V|}{2} + 1$ levels since the same argument works with more levels. Then the average number of nodes at each level is given by $\frac{|V|}{\frac{|V|}{2} + 1} < 2$. Hence there must exist some level with only 1 node, call it $v$. Since $T$ was created by BFS and nodes in a given level can only connect to nodes in adjacent levels then disconnecting $v$ will disconnect $s, t$. \qed

5. Consider an undirected graph $G = (V, E)$ such that every edge of $E$ is colored either red or blue. We redefine the length of a path as the number of blue edges in it. For example, two vertices with a red path between them are at a distance of 0 from each other. Perform a small change in the BFS algorithm, so that it would work according to this new definition of distance.

**Algorithm:**
1) Create two queues. One for blue edges, one for red edges. 2) Pick a starting node to run BFS 3) Run normal BFS except to the following: i) Every time you encounter edges, place them in their appropriate queue ii) Run through all the edges in the red queue first and label the distance as the distance of the parent iii) Once no more red edges are left, run through one blue edge and repeat this process. Augment the distance by 1.

**Correctness:**

**Proof.** Since the red edges do not add new levels to the BFS tree, then we have to run through these edges first but not change their distance. Only once we finish running through red edges should we go to the next level of the tree. We do this by traversing a blue edge, augmenting the distance by 1, and repeating the process. By going over the red edges first, it ensures that we find all nodes that are on the same level before traversing blue edges which will increase the level. \qed