1. Let $G$ be a group and let $H$ be a non-empty subset of $G$, such that for every $a, b \in H$ we have $ab^{-1} \in H$. Prove that $H$ is a subgroup of $G$.

2. Let $G$ be a permutation group of a set $X$, such that $|X| - 2 \geq |G| \geq 2$ and that there are exactly two distinct orbits. Prove that there exists a permutation $g \in G$ that does not contain any cycles of length one in its cycle structure.

3. We have a regular three-dimensional octahedron $O$ (that is, the Platonic solid composed of eight equilateral triangles, four of which meet at each vertex), and three colors. After making $O$ stand vertically on one of its vertices $v$, we partition its faces into four triangles that compose the “top” of $O$, and four triangles that compose the “bottom” of $O$. Let $u$ be the vertex at the top. We wish to color each of the eight triangles. Two colorings of $O$ are considered identical if one can be obtained from the other by rotating $O$ along the line between $v$ and $u$ and possibly also switching the top and the bottom (so that $u$ and $v$ switch). We are not allowed to rotated $O$ such that $v$ is neither the bottom vertex nor the top one. Use the method that we saw in Lecture 21 to count how many distinct colorings of $O$ exist.

4. Consider the sequence with the initial conditions $a_0 = 4$, $a_1 = 16$, and the recurrence relation $a_{i+2} = a_{i+1}^{1/3} a_{i}^{2/3}$. Use generating functions to find the value of $a_n$. Specifically, use the techniques that we saw in Lecture 23 and write down the steps of your calculation. Do not use a computer, although you may skip uninteresting technical steps in your writing (like the detailed steps of solving a set of linear equations). Do not use any techniques that were learned after Lecture 23.

5. Let $b_n$ denote the number of sequences of ones and zeros that are of length $n$ and have the ones only occurring in groups of three or more. Find a recurrence relation for $b_n$ such that the number of elements in this relation does not depend on $n$ (for example, in the recurrence relation that we derived for the Catalan numbers, the number of elements did change according to $n$). Prove your answer.

6. For a positive integer $n$, consider the triangular integer lattice $\{(x, y) \in \mathbb{N} : 0 \leq x \leq y \leq n\}$. We travel the lattice from the point $(0, 0)$ to the point $(n, n)$, such that at each step we either go one step to the right, one step up, or one step diagonally right and up (that is, we either increase the $x$-coordinate by one, increase the $y$-coordinate by one, or increase both coordinates by one). The path does not leave the triangular lattice at any point. Let $c_n$ be the number of such paths.

Find a recurrence relation for $c_n$. Unlike the previous question, the number of elements in this relation may depend on $n$. (Hint: Find the first point on the path that is on the main diagonal $x = y$.) Prove your answer.