1. For integers $0 < r < n$, express $\sum_{k=r}^{n} \binom{k}{r}$ as a single binomial coefficient. Explain your answer.

2. Basic counting:
   (a) For a positive integer $n$ satisfying $n \equiv 1 \mod 4$, how many subsets $S \subseteq \{1, 2, 3, \ldots, n\}$ satisfy that the sum of the numbers in $S$ is larger than the sum of the numbers not in $S$? Explain your answer.
   (b) For positive integers $n$ and $m$, consider the integer lattice $\{(x, y) \in \mathbb{N} : 0 \leq x \leq n \text{ and } 0 \leq y \leq m\}$. We travel the lattice from the point $(0, 0)$ to the point $(n, m)$, such that at each step we either go one step to the right or one step up (that is, we either increase the $x$-coordinate by one, or we increase the $y$-coordinate by one). How many different paths are available to us? Explain your answer.

3. Prove the formula $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ for all $n \geq 1$ (hint: Start with the formula for $(x+y)^n$ and set $y = 1$).

4. Consider a connected undirected graph $G = (V, E)$ and two vertices $s, t \in V$, such that the shortest path between $s$ and $t$ is of length larger than $|V|/2$. Prove that there exists a vertex $v \in V \setminus \{s, t\}$ such that after removing $v$ from $G$ (together with the adjacent edges) there are no paths between $s$ and $t$ (hint: This question is related to the BFS algorithm).

5. Consider an undirected graph $G = (V, E)$ such that every edge of $E$ is colored either red or blue. We redefine the length of a path as the number of blue edges in it. For example, two vertices with a red path between them are at a distance of 0 from each other. Perform a small change in the BFS algorithm, so that it would work according to this new definition of distance.