Ma/CS 6a
Class 26: More Partitions

By Adam Sheffer

Ma/CS 6b

- **6b** is a direct continuation of **6a**.
- Most of the class is more advanced graph theory. Among the topics:
  - Probabilistic methods.
  - Ramsey Theory
  - Algebraic methods and graphs.
  - Error correcting codes.
- Grading: 70% HW and 30% final exam.
- Weirder jokes and stories!
- Less algorithmic and more combinatorial.
Recall: Partitions of a Positive Integer

- For a positive integer \( n \), we denote by \( p(n) \) the number of ways to write \( n \) as a sum of positive integers.

- Example. We can write \( n = 5 \) as 
  
  \[
  5, \quad 4 + 1, \quad 3 + 2, \quad 3 + 1 + 1, \\
  2 + 2 + 1, \quad 2 + 1 + 1 + 1, \\
  1 + 1 + 1 + 1 + 1.
  \]

  so \( p(5) = 7 \).

  - \( p(20) = 627 \).
  - \( p(100) = 190569292 \).

Recall: Ferrers Diagrams

- **Ferrers diagrams** are a graphic way of representing partitions.

\[
14 = 6 + 4 + 3 + 1
\]

\[
p(4) = 5
\]
The Generating Function

\[ P(x) = p(0) + p(1)x + p(2)x^2 + \cdots \]
\[ = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots) \cdots \]
\[ = \prod_{i=1}^{\infty} (1 - x^i)^{-1}. \]

Intuitive Explanation

- When opening the parentheses, \( p(4) \) is the coefficient of \( x^4 \).

\[
(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \\
(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots) \cdots \\
1 + 1 + 2 \\
(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \\
(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots) \cdots \\
4 \\
(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \\
(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots) \cdots \\
1 + 3
\]
Restricted Partitions #1

- Consider partitions of \( n \) with no more than \( k \) identical parts.
- For example, when \( n = 12 \) and \( k = 2 \):
  - \( 3 + 3 + 3 + 3 \) and \( 4 + 4 + 4 \) are not valid.
  - \( 5 + 5 + 2 \) and \( 2 + 2 + 4 + 4 \) are valid.
- **Problem.** What is the generating function of partitions that have no more than \( k \) identical parts?

Restricted Partitions #1 (cont.)

- **Special case.** Taking \( k = 1 \), we get the generating function for \( p(n \mid \text{each part is distinct}) \):
  \[
  (1 + x)(1 + x^2)(1 + x^3) \cdots
  \]
- What about the case of an arbitrary \( k \)?
  \[
  \prod_{n=1}^{\infty} (1 + x^n + x^{2n} + \cdots + x^{kn}).
  \]
Restricted Partitions #2

- Consider partitions of \( n \) with only odd parts.

- For example, when \( n = 12 \):
  - \( 1 + 1 + 1 + \cdots + 1, 3 + 3 + 3 + 3, 11 + 1, \) etc...

- **Problem.** What is the generating function of partitions with only odd parts?
  \[
  (1 - x)^{-1} (1 - x^3)^{-1} (1 - x^5)^{-1} \cdots
  = \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}.
  \]

Restricted Partitions #3

- Consider partitions of \( n \) with only even parts.

- For example, when \( n = 12 \):
  - \( 10 + 2, 2 + 2 + \cdots + 2, 4 + 4 + 4, \) etc...

- **Problem.** What is the generating function of partitions with only even parts?
  \[
  (1 - x^2)^{-1} (1 - x^4)^{-1} (1 - x^6)^{-1} \cdots
  = \prod_{n=1}^{\infty} (1 - x^{2n})^{-1}.
  \]
Restricted Partitions #4

- Consider partitions of \( n \) with each part equals to at most \( k \).
- For example, when \( n = 12 \) and \( k = 4 \):
  - \( 5 + 5 + 2 \) and \( 10 + 1 + 1 \) are not valid.
- **Problem.** What is the generating function of partitions whose parts equal to at most \( k \)?
  \[
  (1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} \cdots (1 - x^k)^{-1} = \prod_{n=1}^{k} (1 - x^n)^{-1}.
  \]

Are These the Same? #1

- The generating function of \( p(n \mid \text{each part is odd}) \) is
  \[
  \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}.
  \]
- The generating function of \( p(n \mid \text{each part is even}) \) is
  \[
  \prod_{n=1}^{\infty} (1 - x^{2n})^{-1}.
  \]
- Does \( p(n \mid \text{each part is odd}) = p(n \mid \text{each part is even}) \) for every \( n \)?
Answer

- No!
  - For example, when \( n \) is odd the number of even partitions is zero and the number of odd partitions is not.

Are These the Same? #2

- The generating function of \( p(n \mid \text{each part is odd}) \) is
  \[
  \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}.
  \]
- The generating function of \( p(n \mid \text{each part is distinct}) \) is
  \[
  \prod_{n=1}^{\infty} (1 + x^n).
  \]
- Does \( p(n \mid \text{each part is odd}) = p(n \mid \text{each part is distinct}) \) for every \( n \)?
Answer

- We now prove that
  \[\prod_{n=1}^\infty (1 - x^{2n-1})^{-1} = \prod_{n=1}^\infty (1 + x^n).\]

- **Proof.** Since \(1 + y = (1 - y^2)/(1 - y),\) we have
  \[\prod_{n=1}^\infty (1 + x^n) = \frac{\prod_{n=1}^\infty (1 - x^{2n})}{\prod_{n=1}^\infty (1 - x^n)} = \prod_{n=1}^\infty (1 - x^{2n-1})^{-1}\]

Odd VS Even Number of Parts

- We define
  \[e_n = p(n \mid \text{distinct parts and their # is even})\]
  \[o_n = p(n \mid \text{distinct parts and their # is odd}).\]

- **Problem.** What is \(e_n - o_n?\)
  - When \(n = 2, e_n - o_n = 0 - 1 = -1.\)
  - When \(n = 3, e_n - o_n = 1 - 1 = 0.\)
  - When \(n = 4, e_n - o_n = 1 - 1 = 0.\)
  - When \(n = 5, e_n - o_n = 2 - 1 = 1.\)
  - When \(n = 6, e_n - o_n = 2 - 2 = 0.\)
The Generating Function

- Set \( q(n) = e_n - o_n \), and
  \[ Q(x) = 1 + q(1)x + q(2)x^2 + q(3)x^3 + \ldots \]

- Claim. \( Q(x) = (1 - x)(1 - x^2)(1 - x^3) \ldots \).

- Proof.
  - A partition \( n = s_1 + \ldots + s_k \) (where the parts are distinct), corresponds to
    \( (-x^{s_1})(-x^{s_2})\ldots(-x^{s_k}) \).
  - That is, every even partition of \( n \) contributes \( x^n \) and every odd partition contributes \( -x^n \).

The Correct Bound

- Theorem. We have
  \[ e_n - o_n = \begin{cases} (-1)^m, & \text{if } n = \frac{1}{2} m(3m \pm 1), \\ 0, & \text{otherwise.} \end{cases} \]

- Proof.
  - Refer to partitions that are counted in \( e_n \) as even, and partitions that are counted in \( o_n \) as odd.
  - We define a map from even to odd partitions, or from odd to even partitions, which is almost a bijection.
Preparation for the Map

- $\lambda$ – a partition with distinct parts.
- $s(\lambda)$ – the size of the smallest part of $\lambda$.
- $t(\lambda)$ – the length of the sequence that starts with the first part of $\lambda$ and continues as long as parts decrease by 1 at each step.

$$s(\lambda) = 1 \quad t(\lambda) = 3$$

The Mapping – Case 1

- If $s(\lambda) \leq t(\lambda)$, we remove the last part of $\lambda$ and add 1 to each of the first $s(\lambda)$ parts.

- This map takes an even partition to an odd partition (or vice versa).
The Mapping – Case 2

- If \( s(\lambda) > t(\lambda) \), we remove 1 from each of the \( t(\lambda) \) largest parts, and add a new smallest part of size \( t(\lambda) \).

- Once again, the map takes an even partition to an odd partition (or vice versa).

Examining Case 1

- When does the mapping of case 1 fails? (assuming that \( s(\lambda) \leq t(\lambda) \)).

- When \( s(\lambda) = t(\lambda) \) = the number of parts of \( \lambda \).
Case 1 is “Almost” Well Behaved

- For what values of \( n \) can the problem in case 1 occur?
  - Write the number of parts as \( m = s(\lambda) = t(\lambda) \).
    
    \[
    n = m + (m + 1) + (m + 2) + \cdots + (2m - 1) = \frac{1}{2} m(3m - 1).
    \]
  - That is, when \( n = \frac{1}{2} m(3m - 1) \) (for some \( m \in \mathbb{N} \)) case 1 fails to map one partition.

Examining Case 2

- When does the mapping of case 2 fails?
  (assuming that \( s(\lambda) > t(\lambda) \)).

- When \( s(\lambda) - 1 = t(\lambda) = \) number of parts.

Case 2 is “Almost” Well Behaved

• For what values of $n$ can the problem in case 2 occur?
  ◦ The number of parts: $m = s(\lambda) - 1 = t(\lambda)$.
    $$n = (m + 1) + (m + 2) + \cdots + 2m$$
    $$= \frac{1}{2} m(3m + 1).$$

• That is, when $n = \frac{1}{2} m(3m + 1)$ (for some $m \in \mathbb{N}$) case 2 fails to map one partition.

Concluding the Proof

• If $n \neq \frac{1}{2} m(3m \pm 1)$, then the mapping is a bijection and thus $e_n - o_n = 0$.

• If $n = \frac{1}{2} m(3m \pm 1)$, then
  ◦ If $m$ is even, the mapping takes the even partitions to distinct odd partitions, with one exception, so $e_n - o_n = 1$.
  ◦ If $m$ is odd, the mapping takes the odd partitions to distinct even partitions, with one exception, so $e_n - o_n = -1$. 
A Simple Observation

- **Recall.**
  - The partitions generating function is
    \[ P(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1}. \]
  - The generating function of \( e_n - o_n \) is
    \[ Q(x) = \prod_{n=1}^{\infty} (1 - x^n). \]

- We thus have
  \[ P(x) \cdot Q(x) = 1. \]

Consequences of the Observation

- We have \( P(x) \cdot Q(x) = 1 \). Equivalently,
  \[ (1 + p(1)x + p(2)x^2 + \ldots)(1 - x - x^2) \]
Computing $p(n)$

- The above technique gives us an efficient recursive method for computing $p(n)$.

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<th>9</th>
<th>10</th>
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The End: Teaching Survey

In the space below please write any comments about physical facilities or use of equipment and technology:

If I had one hour to live, I'd spend it in this class because it feels like an eternity.

Summary Question

19. Overall, how would you rate your learning experience in this course?

In the space below please write any overall comments about this course or instructor not covered above:

Additional Questions (if separate sheet is provided)