Ma/CS 6a
Class 25: Partitions

Explain the significance of the following sequence: un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu...

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Answer

These are the Catalan numbers!

(The numbers one to ten in Catalan.)
Partitions of a Positive Integer

- For a positive integer \( n \), we denote by \( p(n) \) the number of ways to write \( n \) as a sum of (unordered) positive integers.
- Example. We can write \( n = 5 \) as
  
  \[
  5, \quad 4 + 1, \quad 3 + 2, \quad 3 + 1 + 1, \\
  2 + 2 + 1, \quad 2 + 1 + 1 + 1, \\
  1 + 1 + 1 + 1 + 1.
  \]

  so \( p(5) = 7 \).
- \( p(20) = 627 \).
- \( p(100) = 190569292 \).

Ferrers Diagrams

- *Ferrers diagrams* are a graphic way of representing partitions.

\[
14 = 6 + 4 + 3 + 1
\]

\[
p(4) = 5
\]
A Simple Observation

• **Claim.** Let $n$ and $r$ be positive integers. Then
  
  $$p(n \mid \text{number of parts } \leq r) = p(n + r \mid \text{number of parts } = r).$$

• **Proof.** We find a **bijection** between the two sets of partitions:

\[
\begin{align*}
  n + r & \quad \leftrightarrow \quad n \\
  \circ \circ \circ \circ \circ \circ & \quad \leftrightarrow \quad \circ \circ \circ \circ \circ \circ
\end{align*}
\]
Detailed Proof

- We describe a bijection between the sets:
  - $P_n$ - Partitions of $n$ with at most $r$ parts.
  - $P_{n+r}$ - Partitions of $n + r$ with exactly $r$ parts.

- Given a partition of $P_n$, we add a new first column with $r$ elements, obtaining a partition of $P_{n+r}$.
- Given a partition of $P_{n+r}$, we remove the first column to obtain a partition of $P_n$.

Conjugate Partitions

- Two partitions of a number $n$ are said to be conjugate if one is obtained from the other by switching the rows and columns in the Ferrers Diagram.
Using Conjugate Partitions

- Consider a pair of conjugate partitions \( \alpha, \beta \). The **size of the largest part of** \( \alpha \) **is** the **number of parts of** \( \beta \).
- Using a **bijection** argument as before, we have
  \[
  p(n \mid \text{largest part of size } m) = p(n \mid \text{number of parts } = m).
  \]

Self-Conjugation

- A partition is **self-conjugate** if it is its own conjugate.
- **Claim.**
  \[
  p(n \mid \text{self-conjugate}) = p(n \mid \text{the parts are distinct and odd}).
  \]
Self Conjugation Proof

\[ p(n \mid \text{self-conjugate}) = p(n \mid \text{the parts are distinct and odd}). \]

- **Proof.** As before, we find a bijection between the two sets of partitions.
- Given a self conjugate partition, let \( k_i \) be the number of elements in the 1st row and column after removing the first \( i - 1 \) rows and columns. For \( i < j \), we have \( k_i > k_j \).
- We use the \( 2k_i - 1 \) elements in the \( i' \)th “row and column” to create the \( i' \)th row.

Partitions and Generating Functions

- To calculate \( p(i) \), we define a generating function for the number of partitions:
  \[ P(x) = p(0) + p(1)x + p(2)x^2 + \cdots \]
  - By convention, we write \( p(0) = 1 \).
  
- We have as many initial values as we like:
  - \( p(1) = 1, \ p(2) = 2, \ p(3) = 3, \ p(4) = 5, \ p(5) = 7, \ldots \)

- Not clear how to find a recursive relation.
Warm-Up Question

• For any positive integer \( n \), we have
\[
(1 - x^n)^{-1} = 1 + x^n + x^{2n} + x^{3n} + \ldots
\]

• Let \( p_n(m) \) denote the number of partitions of \( i \) where each part is of size \( n \).
\[
p_n(m) = \begin{cases} 
1, & \text{if } n|m, \\
0, & \text{otherwise.}
\end{cases}
\]

• The corresponding generating function:
\[
P_n(x) = p_n(0) + p_n(1)x + p_n(2)x^2 + \ldots
= 1 + x^n + x^{2n} + x^{3n} + \ldots = (1 - x^n)^{-1}.
\]

A Bit of Progress

• Let \( p_{n,m}(i) \) denote the number of partitions of \( i \) where each part is equal to either \( m \) or \( n \).

• Let
\[
P_{n,m}(x) = p_{n,m}(0) + p_{n,m}(1)x + p_{n,m}(2)x^2 + \ldots
= (1 + x^n + x^{2n} + \ldots)(1 + x^m + x^{2m} + \ldots)
= (1 - x^n)^{-1}(1 - x^m)^{-1}.
\]
**Intuitive Explanation**

- When opening the parentheses, \( p_{2,4}(10) \) is the coefficient of \( x^{10} \).

\[
(1 + x^2 + x^4 + x^6 + x^8 + \cdots)(1 + x^4 + x^8 + \cdots)
\]

\[
2 + 4 + 4
\]

\[
(1 + x^2 + x^4 + x^6 + x^8 + \cdots)(1 + x^4 + x^8 + \cdots)
\]

\[
2 + 2 + 2 + 4
\]

\[
(1 + x^2 + x^4 + x^6 + x^8 + x^{10} + \cdots)(1 + x^4 + \cdots)
\]

\[
2 + 2 + 2 + 2 + 2
\]

**Changing a Dollar**

- **Problem.** In how many ways can a dollar be exchanged for quarters (25c), dimes (10c), and nickels (5c)?

- To make the numbers simpler, we can divide everything by 5:
  - In how many ways can we write 20 as a sum of 1’s, 2’s, and 5’s.
  - The coefficient of \( y^{20} \) in \( (1 - y)^{-1}(1 - y^2)^{-1}(1 - y^5)^{-1} \).
Number Crunching

• First, let us calculate

\[
(1 - y^2)^{-1}(1 - y^5)^{-1} = (1 + y^2 + y^4 + \cdots + y^{20})(1 + y^5 + y^{10} + y^{15})
\]

Number Crunching (cont.)

• We have

\[
(1 - x^2)^{-1}(1 - x^5)^{-1} = 1 + y^2 + y^4 + y^5 + y^6 + y^7 + y^8 + y^9 + 2y^{10} + y^{11} + 2y^{12} + y^{13} + 2y^{14} + 2y^{15} + 2y^{16} + 2y^{17} + 2y^{18} + 2y^{19} + 3y^{20}.
\]

• What is the coefficient of \(y^{20}\) in

\[
(1 - y)^{-1}(1 - y^2)^{-1}(1 - y^5)^{-1}?
\]

◦ Every element of \((1 - y^2)^{-1}(1 - y^5)^{-1}\) corresponds to one way of writing 20:

\[
1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 + 1 + 2 + 1 + 2 + 1 + 2 + 2 + 2 + 2 + 2 + 2 + 3 = 29
\]
Back to General Partitions

**Theorem.** The generating function of \( p(n) \) can be written as

\[
P(x) = p(0) + p(1)x + p(2)x^2 + \cdots = \prod_{i=1}^{\infty} (1 - x^i)^{-1} = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3)
\]

**Proof Sketch**

- We need to verify that the coefficient of \( x^n \) in \( P(x) \) is \( p(n) \).
  - **Consider a partition** \( n = m_1s_1 + m_2s_2 + \cdots + m ks_k \), where \( s_1, \ldots, s_k \) are distinct numbers and \( m_i \) is the number of parts of size \( s_i \) in the partition.
  - In \( \prod_{i=1}^{\infty} (1 - x^i)^{-1} \), this partition corresponds to taking \( x^{m_is_i} \) from \( (1 + x^{s_1} + x^{2s_1} + \cdots) \).
  - Similarly, **any choice of elements from** the parentheses in \( \prod_{i=1}^{\infty} (1 - x^i)^{-1} \) that yields \( x^n \) corresponds to a partition of \( n \).
A Small Issue

- Our proof is fine if we have a product of finitely many terms, but in 
  \[ \prod_{i=1}^{\infty} (1 - x^i)^{-1} \] we have products of infinitely many terms!
  - When proving that the coefficient of \( x^n \) is \( p(n) \), it suffices to consider \( \prod_{i=1}^{n} (1 - x^i)^{-1} \).

Restricted Partitions #1

- Consider partitions of \( n \) with no more than \( k \) identical parts.
- For example, when \( n = 12 \) and \( k = 2 \):
  - \( 3 + 3 + 3 + 3 \) and \( 4 + 4 + 4 \) are not valid.
  - \( 5 + 5 + 2 \) and \( 2 + 2 + 4 + 4 \) are valid.
- **Problem.** What is the generating function of partitions that have no more than \( k \) identical parts?
Restricted Partitions #1 (cont.)

- **Special case.** Taking $k = 1$, we get the generating function for $p(n |$ each part is distinct): 
  \[(1 + x)(1 + x^2)(1 + x^3) \cdots\]

- What about the case of an arbitrary $k$?

\[
\prod_{n=1}^{\infty} \left(1 + x^n + x^{2n} + \cdots + x^{kn}\right).
\]

Restricted Partitions #2

- Consider partitions of $n$ with only odd parts.

- For example, when $n = 12$:
  \[
  1 + 1 + 1 + \cdots + 1, 3 + 3 + 3 + 3, 11 + 1, \text{ etc...}
  \]

- **Problem.** What is the generating function of partitions with only odd parts?
  \[
  (1 - x)^{-1}(1 - x^3)^{-1}(1 - x^5)^{-1} \cdots
  \]
  \[
  = \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1}.
  \]
Restricted Partitions #3

- Consider partitions of \( n \) with only even parts.
- For example, when \( n = 12 \):
  - \( 10 + 2, 2 + 2 + \cdots + 2, 4 + 4 + 4, \) etc...
- **Problem.** What is the generating function of partitions with only even parts?
  \[
  (1 - x^2)^{-1}(1 - x^4)^{-1}(1 - x^6)^{-1} \cdots = \prod_{n=1}^{\infty} (1 - x^{2n})^{-1}
  \]

Restricted Partitions #4

- Consider partitions of \( n \) with each part equals to at most \( k \).
- For example, when \( n = 12 \) and \( k = 4 \):
  - \( 5 + 5 + 2 \) and \( 10 + 1 + 1 \) are not valid.
- **Problem.** What is the generating function of partitions whose parts equal to at most \( k \)?
  \[
  (1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} \cdots (1 - x^k)^{-1} = \prod_{n=1}^{k} (1 - x^n)^{-1}
  \]
Happy Thanksgiving!