Ma/CS 6a
Class 13: Network Flow

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The RAND Corporation

- American think tank composed of scientists.
- In the 50’s helped decision concerning the nuclear race, space program, etc.
- Contributed to the development of many scientific techniques, such as game theory.

John Nash  John von Neumann  Dr. Strangelove
A document that was declassified in 1999 describes how RAND studied the Soviet train system.

They studied the Soviet ability to transport things from place to place (e.g., Asian side to European side).

**Recall: Directed Graphs**

- In a **directed graph** (or **digraph**), every edge has a direction.
- A **directed path** is a path that follows the direction of the edges.

✓ $a \to d \to c \to b$

✗ $a \to d \to e \to c \to b$
Flow Networks

- A **flow network** is a digraph $G = (V, E)$, together with a **source** vertex $s \in V$, a **sink** vertex $t \in V$, and a **capacity function** $c: E \to \mathbb{N}$.

![Graph Image]

Flow in a Network

- Given a flow network $G = (V, E, s, t, c)$, a **flow** in $G$ is a function $f: E \to \mathbb{N}$ that satisfies
  - Every $e \in E$ satisfies $f(e) \leq c(e)$.
  - Every $v \in V \setminus \{s, t\}$ satisfies
    $$\sum_{(u,v)\in E} f(u,v) = \sum_{(v,w)\in E} f(v,w)$$
    **Total flow entering $v$.**
    **Total flow exiting $v$.**
Example: Flow

- The capacities are in red.
- The flow is in blue.

What is it Good For?

Pipe Network analysis

Ecological food webs

Transportation networks
Size of a Flow

- In any flow $f$, the flow coming out of $s$ is equal to the flow getting into $t$, since nothing is allowed to accumulate at intermediate vertices.
- We refer to this quantity as the size of $f$ (or $|f|$).

Maximum Flow

- We are usually interested in finding the maximum flow of a network.
- We wish to have an algorithm that receives a flow network, and finds a maximum flow of it.
An Initial Flow

- Find a path from $s$ to $t$.
  - For example, the path $s \rightarrow b \rightarrow d \rightarrow t$.
  - The flow that can pass in this path is the \textbf{minimum edge capacity} - 2.

Increasing the Flow

- Find another path from $s$ to $t$, among the edges that are not \textit{saturated} ($f(e) \neq c(e)$).
  - For example, the path $s \rightarrow a \rightarrow t$.
  - The flow that can pass in it is the minimum edge capacity – 3.
Maximum Flow?

- We keep increasing the flow until there are no more paths of unsaturated edges.
- Does that mean that we have a maximum flow?
  - No! This network has a flow of size 10.

A Different Kind of Path

- We can go in the opposite direction of an edge $e$, if $f(e) > 0$.
- $s \to a \to d \to c \to t$. 
Residual Network

- Given a flow network \((V, E, s, t, c)\), and a flow \(f\), the residual network of \(f\) is a network with
  - The same set of vertices \(V\), source \(s\), and sink \(t\).
  - Every edge \(e \in E\) has the new capacity \(c'(e) = c(e) - f(e)\).
  - For every edge \((u, v) \in E\), we add the edge \((v, u)\) with capacity \(f(e)\).

Example: Residual Network

A network and a flow

The residual network
The Ford-Fulkerson Algorithm

- Start with a flow \( f \) of size zero.
- Repeat:
  - Build the residual network of \( f \).
  - Find a path \( P \) from \( s \) to \( t \) in the residual network. If there is no such path, stop and return \( f \).
  - Set \( d = \min_{e \in P} c(e) \).
  - Increase the flow in \( f \) of every edge in \( P \) by \( d \) (for reverse edges, decrease the flow by \( d \)).

Not Efficient

- The algorithm can have a **VERY** large running time:

\[ \begin{array}{ccc}
   & a & \\
 s & 0/10^{10} & 1/10^{10} \\
 & 1/10^{10} & 0/10^{10} \\
 & 1/10^{10} & 2/10^{10} \\
c & 1/10^{10} & 1/10^{10} \\
& 0/10^{10} & 1/10^{10} \\
t & 1/10^{10} & 1/10^{10} \\
\end{array} \]
The Running Time

- What is an upper bound for the number of steps of the Ford-Fulkerson algorithm?
  - The size of the maximum flow $|f|$.
  - Running time of the algorithm: at most $c|f|(|V| + |E|)$.

- We can improve the running time by using BFS to find a path in every residual network.
  - This gives a running time of $c|V||E|^2$.
  - Not in the material of this course!
A Recent Development

• In 2012, an algorithm with a running time of $c|V||E|$ was discovered.

• Combining the results of two groups:

J. Orlin

V. King

S. Rao

R. Tarjan

Irrational Capacities

$$r = \frac{-1 + \sqrt{5}}{2} \approx .618...$$

• The missing capacities are all $r + 2$.

• Even though the size of the maximum flow is 2, Ford-Fulkerson might run forever!
Bounding the Size of the Maximum Flow

• Given a flow network, what is an easy upper bound for the size of the maximum flow?

\[ |F| \leq \min \left\{ \sum_{(s,v) \in E} c(s,v), \sum_{(u,t) \in E} c(u,t) \right\}. \]

Cuts

• A cut is a partitioning of the vertices of the flow network into two sets \( S, T \) such that \( s \in S \) and \( t \in T \).

• The size of a cut is the sum of the capacities of the edges from \( S \) to \( T \).
Cuts and Maximum Flow

- If there exists a cut of size $d$, no flow can be of size more than $d$.
  - Each part of the flow must pass through the cut.
  - At most $d$ can pass through the cut.

Minimum Cuts

- A minimum cut is a cut of minimum size in the flow network.
  - Is the cut in the figure a minimum cut?
  - No! There is a cut of size 6.
Max Flow – Min Cut

- We saw that the size of the *minimum cut* \( \geq \) *size of maximum flow*.

- **Max flow – min cut theorem.** In every flow network, the size of the minimum cut *is equal* to the size of the maximum flow.

Proof: Max Flow – Min Cut

- Let \( f \) be a *maximum flow* of the network.
- In the *residual network* \( R \) of \( f \), there is no path from \( s \) to \( t \).
- Let \( S \) be the set of vertices that are accessible from \( s \) in \( R \).
- Let \( T = V \setminus S \). Then \( (S, T) \) is a cut.
The Cut \((S,T)\): Claim 1

- **Claim 1.** An edge \((u, v)\) such that \(u \in S\) and \(v \in T\) must be saturated in \(f\).
  - Since \(u \in S\), there is a path from \(s\) to \(u\) in \(R\).
  - If \((u, v)\) is not saturated, there is a path from \(s\) to \(v\) in \(R\). **Contradicting that \(v \in T\)!**

The Cut \((S,T)\): Claim 2

- **Claim 2.** An edge \((u, v)\) such that \(u \in T\) and \(v \in S\) cannot have flow through it in \(f\).
  - Since \(v \in S\), there is a path from \(s\) to \(v\) in \(R\).
  - If \(f(u, v) > 0\), there is a path from \(s\) to \(u\) in \(R\). **Contradicting that \(u \in T\).**
Completing the Proof

- $f$ - maximum flow.
- $(S, T)$ - a cut defined according to $f$.
  - Every edge from $S$ to $T$ is saturated in $f$.
  - Every edge $e$ from $T$ to $S$ satisfies $f(e) = 0$.
- Thus, the size of the cut $(S, T)$ is $|f|$.
- Thus, the minimum cut has size at most $|f|$.
- That is, the minimum cut is of size $\leq \text{max flow}$.

Cut Size = Flow Size

- **Claim.** Consider a flow network with a flow $f$ and a cut $(S, T)$, both of the same size. Then $f$ is a max flow and $(S, T)$ is a min cut.
- **Proof.**
  - There cannot exist a larger flow, since no flow can be larger than the size of $(S, T)$.
  - There cannot be a cut of size smaller than $|f|$. 
Ford-Fulkerson Correctness

- **Claim.** A flow $f$ is a maximum flow if and only if there are no $s$-$t$ paths in the residual network $R$ of $f$.

- $\Rightarrow$: If there is an $s$-$t$ path in $R$, we can increase $f$, so $f$ is not a maximum flow.

- $\Leftarrow$: If there are no $s$-$t$ paths in $R$, we can define a cut of size $|f|$ as before. This implies that $|f|$ is the size of the min cut, so $f$ is a maximum flow.

The End

- In the Vietnam war, the Vietcong used a series of underground tunnels called the **Ho Chi Minh trail**.

- The US was looking for the min cut to efficiently disconnect the south and the north parts of the system.

- The capacity of an edge is the difficulty of destroying it.