Ma/CS 6a
Class 5: Basic Counting

Permutations

- **Problem.** Given a set \{1,2, ..., n\}, in how many ways can we order it?
- **The case** \(n = 3\). Six distinct orders / permutations: 123, 132, 213, 231, 312, 321.
- **The general case.**

\[ n! = n \cdot (n - 1) \cdot \ldots \cdot 2 \cdot 1 \]
Total Number of Subsets

• **Problem.** How many subsets does the set $S = \{1, 2, \ldots, n\}$ have?
  ◦ Two options for every element $i \in S$. Either $i$ is in the subset or not.
  ◦ Since there are $n$ element in $S$, the number of subsets is $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$.

Subsets of Size $k$

• Given a set $\{1, 2, \ldots, n\}$, how many (unordered) **subsets of size $k$** does it have?

• **Example.** Consider the case $n = 5$ and $k = 3$.
  ◦ The possible subsets are $(1,2,3), (1,2,4), (1,2,5), (1,3,4), (1,3,5), (1,4,5), (2,3,4), (2,3,5), (2,4,5), (3,4,5)$.
  ◦ **10 distinct subsets!**
Subsets of Size $k$ (cont.)

- Given a set $S = \{1, 2, \ldots, n\}$, how many (unordered) subsets of size $k$ does it have?
- Look at the $n!$ orderings of $S$ and consider the first $k$ numbers as the subset.
  - For example, when $n = 5$ and $k = 3$
    - $12345$  $34251$
    - $13524$  $34152$
    - $54321$  $13542$

Binomial Coefficients

- Given a set $S = \{1, 2, \ldots, n\}$, how many (unordered) subsets of size $k$ does it have?
- Look at the $n!$ orderings of $S$ and consider the first $k$ numbers as the subset.
  - Every subset is obtained $k! \frac{(n-k)!}{(n-k)!}$ times, so
\[
\binom{n}{k} = \frac{n!}{k! (n-k)!}
\]

Pronounced “$n$ choose $k$”
Warm-up Problem

- **Prove or disprove.** For every \( n \geq k \geq 0 \)

\[
\binom{n}{k} = \binom{n}{n-k}.
\]

- **True.** Deciding which \( k \) elements to choose is like deciding which \( n - k \) elements not to take.

Pascal’s Rule

- **Prove.** For every \( n \geq k \geq 0 \)

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

# of subsets containing 1

# of subsets not containing 1
Pascal’s Triangle

- **Pascal’s rule:** \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \).
- \( \binom{n}{k} \) is element \( k + 1 \) of row \( n + 1 \).

\[
\begin{array}{ccccccc}
& & & & & & 1 \\
& & & & & 1 & 1 \\
& & & & 1 & 2 & 1 \\
& & & 1 & 3 & 3 & 1 \\
& & 1 & 4 & 6 & 4 & 1 \\
& 1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

*Every number is the sum of the two numbers above it.*

A Sum of Binomial Coefficients

- **Prove.** For every \( n \geq k \geq 0 \)
  \[
  \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.
  \]
- The left-hand side is the number of subsets of \( \{1, 2, 3, \ldots, n\} \), which is \( 2^n \).
Partitioning into $k$ Subsets

- **Problem.** For $n, k > 0$, we have $n$ identical balls and $k$ bins. In how many ways can we place the balls in the bins?

- **Example.** If we have three balls and two bins, there are four options: $(3,0)$, $(2,1)$, $(1,2)$, $(0,3)$.

- **Answer.** $\binom{n + k - 1}{k - 1}$. The $k - 1$ choices correspond to the end of each bin.

Bin #1: 1 ball  
Bin #2: 3 balls  
Bin #4: empty
The Binomial Theorem

• Recall.
  ◦ $(x + y)^2 = x^2 + 2xy + y^2$.
  ◦ $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

• The binomial theorem. What is $(x + y)^n$?

\[
\sum_{0 \leq i, j \leq n, \ i + j = n} \binom{n}{i} x^i y^j
\]

\[
= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots
\]

The Binomial Theorem – Proof

• The binomial theorem.

\[
(x + y)^n = \sum_{0 \leq i \leq n} \binom{n}{i} x^i y^{n-i}.
\]

• Proof. We have

\[
(x + y)^n = (x + y)(x + y) \cdots (x + y).
\]

• The coefficient of $x^i y^{n-i}$ is the number of ways to choose $x$ from $i$ of the parentheses and $y$ from the remaining ones.

• That is, the coefficient of $x^i y^{n-i}$ is $\binom{n}{i}$.
The Binomial Theorem and Pascal’s Triangle

\[(x + y)^1 = x + y\]
\[(x + y)^2 = x^2 + 2xy + y^2\]
\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\]
\[(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]
\[\ldots\]

```
    1
   1 1
  1 2 1
 1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
```

Monomials and Degrees

- Polynomials are sums of monomials:
  \[x^7 + 3x^2y^4z + 5x^3z^3 + \ldots\]

- The degree of a monomial is the sum of the powers of its variables.
  \[\deg(3x^2y^4z) = 2 + 4 + 1 = 7.\]

- The degree of a polynomial is the maximum of the degrees of its monomials
  \[\deg(x^5 + 3x^2y^4z + 5x^3z^3) = 7\]
Number of Monomials

- **Problem.** How many distinct monomials can a polynomial of degree $D$ in $k$ variables have?

- **Answer.** Take $k + 1$ bins – one for every variable and one extra. Every placement of $D$ balls in the bins corresponds to a monomial.

\[
\binom{D + k}{k}
\]
Returning to Lecture 3

- To prove Fermat’s little theorem we assumed, without proof, that for any prime \( p \)
  \[(a + b)^p \equiv a^p + b^p \mod p.\]

- **Proof.** By the binomial theorem:
  \[(a + b)^p = \binom{p}{0} a^p + \binom{p}{1} a^{p-1} b + \binom{p}{2} a^{p-2} b^2 + \cdots\]

- To prove the claim, it suffices to prove that
  \( p \mid \binom{p}{i} \) for every \( 1 \leq i \leq p - 1 \).

- This holds since in \( \binom{p}{i} = \frac{p!}{i!(p-i)!} \) the numerator is divisible by \( p \) but the denominator is not.

Partitions of an Integer

- \( r, n \) – two positive integers.

- **Problem.** What is the number of solutions of
  \[ a_1 + a_2 + \cdots + a_r = n, \]
  where each \( a_i \) is a natural number?

  \[ 5 = 1 + 1 + 3 = 1 + 3 + 1 = 0 + 0 + 5 = 1 + 0 + 4 = \cdots \]
Solution

• Consider \( n \) as a sum of \( n \) unit elements.
• Dividing these elements across the \( r \) variables \( a_i \) is equivalent to placing \( n \) balls in \( r \) bins.
  ◦ The value of \( a_i \) is the number of balls in the \( i \)'th bin.

\[
\binom{n+r-1}{r-1}
\]

Another Inequality

• **Problem.** Prove the identity

\[
\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.
\]

• **Proof.**
  ◦ We begin with the identity

\[
(1 + x)^n(1 + x)^n = (1 + x)^{2n}.
\]
  ◦ By the binomial theorem, we have

\[
\left(\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n\right)\left(\binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n\right).
\]
Proof (cont.)

\[
\left( \binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n} x^n \right) \left( \binom{n}{0} + \binom{n}{1} x + \cdots \right)
\]

Summing Up

- In how many ways can we choose \( k \) elements from \( \{1, 2, 3, \ldots, n\} \)?

<table>
<thead>
<tr>
<th></th>
<th>Ordered</th>
<th>Unordered</th>
</tr>
</thead>
<tbody>
<tr>
<td>No repetitions</td>
<td>( \frac{n!}{(n-k)!} )</td>
<td>( \binom{n}{k} )</td>
</tr>
<tr>
<td>With repetitions</td>
<td>( n^k )</td>
<td>( \binom{k+n-1}{n-1} )</td>
</tr>
</tbody>
</table>
Summing Up #2

- In how many ways can we place \( k \) balls into \( n \) bins?

<table>
<thead>
<tr>
<th></th>
<th>At most 1 ball in each bin</th>
<th>Any number of balls in each bin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Each ball has a different color</td>
<td>( \frac{n!}{(n-k)!} )</td>
<td>( n^k )</td>
</tr>
<tr>
<td>Balls are indistinguishable</td>
<td>( \binom{n}{k} )</td>
<td>( \binom{k+n-1}{n-1} )</td>
</tr>
</tbody>
</table>

The End