Problem 1.

Proof. $g \in C_G(H) \iff g$ commutes with everything in $H$.

Clearly 1 has this property and if $g \in H$, so does $g^{-1}$. Closure under multiplication is also readily verified.

Let’s establish normality. If $a \in G$ is any element, and $h \in H$, then $aga^{-1} \cdot h \cdot ag^{-1}a^{-1} = ag(a^{-1}ha)g^{-1}a^{-1}$.

Since $H$ is normal, $a^{-1}ha := h' \in H$. Then $ag^{-1}h'ag^{-1}a^{-1} = ah'a^{-1}$, since, $g$ and $h'$ commute. Finally $ah'a^{-1}aa^{-1}haa^{-1} = h$. Thus $aga^1$ commutes with everything in $H$. We have shown $g \in C_G(H) \Rightarrow aga^{-1} \in C_G(H)$ for all $a$, hence $C_G(H)$ is a normal subgroup of $G$.

Finally, given $\overline{g} \in G/C_G(H)$, assign to it the automorphism $\phi_g$ of $H$ which sends $h \mapsto ghg^{-1}$. This is well-defined, for if $\overline{g} = \overline{g}'$, then $g^{-1}g \in C_G(H)$, and hence $g'hg^{-1} = ghg^{-1}$, for all $h$.

We need to check that $\phi$ is injective. Indeed, if $\phi_g$ is the identity automorphism of $H$, then $ghg^{-1} = h \iff g$ commutes with everything in $H \iff g \in C_G(H)$, i.e. $\overline{g} = \text{id}$ in $G/C_G(H)$.

Alternatively, consider the group homomorphism $G \rightarrow \text{Aut}(H)$ which sends $g \in G$ to $\phi_g$ above (i.e. $\phi_g(h) = ghg^{-1}$). Then the above paragraph shows that the kernel is $C_G(H)$. Thus $C_G(H)$ is normal in $G$ and the quotient $G/C_G(H)$ isomorphic to a subgroup of $H$.

Problem 2.

Proof. (a) The stabilizer of $x_1 + x_2 + x_3$ is $S_3$. The stabilizer of $x_1x_2 + x_3x_4$ is

$$\{\text{id}, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}.$$

(b) Every permutation in the stabilizer must fix $x_4$. Thus the stabilizer is a subgroup of $S_3$. It contains no transposition, as that changes the sign of the polynomial. From here we easily conclude that the stabilizer is $\{\text{id}, (123), (132)\}$.

(c) The stabilizer has order 8, hence the orbit has cardinality $24/8 = 3$. A quick inspection shows that the orbit is $\{x_1x_2 + x_3x_4, x_1x_3 + x_2x_4, x_1x_4 + x_2x_3\}$.

Problem 3.

Proof. (a) $G$ acts on the $p$ cosets of $H$, hence we have a group homomorphism $G \rightarrow S_p$. Let $K$ be its kernel. Clearly $K$ is contained in $H$ (since $k \in K$ fixes the trivial coset $H$ in particular, and $kH = H$ implies $k \in H$).

Then $G/K$ is isomorphic to a subgroup of $S_p$, so its order divides $p!$. On the other side, the order also divides $|G| = n$. Since $p$ is the smallest prime of $n$, we must have $|G/K| = p$. Hence both $H$ and $K$ have index $p$ in $G$. Since $K \subseteq H$, we must have $K = H$ and thus $H$ is normal.

(b) Assume $G$ is simple. We replicate the above argument. The action of $G$ on the cosets of $H$ induces a homomorphism $G \rightarrow S_d$, which must be trivial or injective, since $G$ is simple. If $G \rightarrow S_d$ is injective, then $|G|$ divides $d!$, a contradiction. If it is trivial, then this forces $H$ to be equal to $G$, i.e. $d = 1$, again a contradiction.
Problem 4.

Proof.  (a) Page 123, proposition 6.

(b) $H$ is a subgroup of $N_G(H)$, hence the number of conjugate subgroups equals $[G : N_G(H)]$ which divides $[G : H]$.

(c) Thus there are at most $[G : H]$ conjugate subgroups, each of size $H$. Thus the union of distinct $gHg^{-1}$'s can have at most $[G : H] \cdot |H| = |G|$ elements, with equality only if the $gHg^{-1}$'s are pairwise disjoint. This is clearly not the case, since each contains the identity for example. Thus $G \neq \bigcup_{g \in G} gHg^{-1}$.

Problem 5.

Proof. $X$ is partitioned into orbits by the action of $G$. The size of each such orbit is divisible by $p$, unless the orbit is a singleton. This follows from the fact that the product of the orbit size and the stabilizer size equals $|G| = p^n$. Since $|X|$ is a multiple of $p$, the number of singleton orbits must thus be a multiple of $p$, so in particular $\geq p$ if it is nonzero.

Problem 6.

Proof.  (a) Let $x \in O_i$ and $gx \in O_j$. We claim that $gO_i = O_j$. Indeed, if $y \in O_i$, then $y = hx$ for some $h \in H$, and hence $gy = ghx = h'gx \in O_j$, since $gx \in O_j$. Note that the existence of $h' \in H$ so that $gh = h'g$ follows from the normality of $H$.

Thus $gO_i \subseteq O_j$. Conversely, if $y \in O_j$, then $y = hgx$ for some $h \in H$ (since $gx \in O_j$ and $O_j$ is the orbit of $gx$ under the action of $H$).

Then $hg = gh'$ for some $h' \in H$ (again by normality of $H$), hence $y = gh'x = g(h'x) \in gO_i$, establishing the reverse inclusion.

(b) Transitivity follows from the fact that the action of $G$ is transitive (given $O_i$ and $O_j$ choose $x \in O_i$, $y \in O_j$ and $g \in G$ such that $gx = y$. Then the above discussion shows that $gO_i = O_j$).

Since $|O_i| = |gO_i|$ for any $g$, $i$ and $G$ acts transitively on $\Sigma$, it follows that $|O_i| = |O_j|$ for all $i, j$.

Consider the action of $H$ on $O_1$. This is transitive (by the definition of orbit), and an element $a \in O_1$ is stabilized (inside $H$) by $H \cap G_a$. From the orbit-stabilizer theorem, it then follows that $|O_1| = [H : H \cap G_a]$.

Problem 7.

Proof.  20 From the previous problem the conjugacy class of $\sigma$ breaks down into $k$ conjugacy classes in $A_n$, where $k = |S_n : A_nC_{S_n}(\sigma)|$. Thus if $\sigma$ commutes with an odd permutation, then $A_nC_{S_n}(\sigma) = S_n$, and $k = 1$. If not, then $A_nC_{S_n}(\sigma) = A_n$ and $k = 2$.

22 Use problem 21 in the book. Alternatively, we want to show that an $n$-cycle does not commute with any odd permutation. For example if $\sigma$ is any permutation, then $(12\ldots n)\sigma$ sends $i \mapsto \sigma_i + 1$ (modulo $n$), while
\[ \sigma(12\ldots n) \text{ sends } i \mapsto \sigma_{i+1}. \] If \((12\ldots n)\) and \(\sigma\) were to commute, this would imply, \(1 + \sigma_i = \sigma_{i+1}\), for all \(i\), forcing \(\sigma\) to be a power of \((12\ldots n)\), which is even.

\[ \square \]

**Problem 8.**

Proof. 34 Every conjugate of \(P\) has \(p\) elements, one of which is the identity. The others are conjugates of \(p\)-cycles, hence are \(p\)-cycles themselves. Using the formula in problem 33, or just by a combinatorial argument, there are \((p-1)!\) cycles of size \(p\) in \(S_p\). Thus there are \((p-1)!/(p-1) = (p-2)!\) conjugates of \(S_p\), and hence \(|N_{S_p}(P)|\) is the order of \(S_p\) divided by the number of conjugates (Proposition 6), hence equals \(p!/(p-2)! = p(p-1)\).

35 An element of order \(p\) is a product of \(r \geq 1\) disjoint \(p\)-cycles. Thus, there are \(\left\lfloor \frac{n}{p} \right\rfloor\) conjugacy classes, corresponding to elements that are a product of \(1\) \(p\)-cycle, \(2\) \(p\)-cycles, \ldots, \(\left\lfloor \frac{n}{p} \right\rfloor\) \(p\)-cycles.

\[ \square \]

**Problem 9.**

Proof. (a) WLOG, assume \(x = (123)\). Replacing \(y\) by \(y^2\) if necessary, we can assume WLOG that \(y = (124)\). Note that \(\langle x, y \rangle\) contains only even permutations, so it is a subgroup of \(A_4\). Then \(xyx = (143)\). So \(\langle x, y \rangle\) contains at least 6 different 3-cycles. Together with the identity, this means at least 7 permutations. Since \(|A_4| = 12\), we must have \(\langle x, y \rangle = A_4\).

(b) If \(x = y\), we clearly get \(\mathbb{Z}/3\mathbb{Z}\). If \(x, y\) are as in (a) and keep 5 fixed, then we get \(A_4\). If \(x\) and \(y\) don’t fix any element, then without loss of generality, suppose that \(x = (abc)\) and \(y = (ade)\). Then we have \((abc)(ade) = (adebc)\) so the subgroup generated by \(x\) and \(y\) contains a 5-cycle. Moreover, it contains a product of two disjoint 2-cycles. Indeed, \((adebc)^2 = (acebd)\) and \((ade)(acebd) = (bd)(ce)\). So the subgroup contains elements of order 2, 3, and 5, so its order has to be divisible by 30 and divide 120. Since clearly no odd permutations are in the subgroup, the sub-group is a subgroup of \(A_5\). So it has to be either \(A_5\) or a subgroup of \(A_5\) of order 30. But no subgroup of \(A_5\) of order 30 exists since otherwise it would be normal in \(A_5\) which contradicts the fact that \(A_5\) is simple.

\[ \square \]