Problem 1.

Proof. $\mathbb{Z}/12\mathbb{Z}$ is abelian, so any subgroup is normal. Then $\mathbb{Z}/12\mathbb{Z} \supset \mathbb{Z}/4\mathbb{Z} \supset \mathbb{Z}/2\mathbb{Z} \supset \{1\}$ is a composition series.

The factors are $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. □

Problem 2.

Proof. You may recall from a previous homework (or midterm) that $K = \{(1), (12)(34), (13)(24), (14)(23)\}$ is isomorphic to the Klein group, and is normal in $A_4$. We thus have

$$S_4 \supset A_4 \supset K \supset \{(1), (12)(34)\} \supset 1.$$ 

The factors are, in order, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. □

Problem 3.

Proof. Note that $A_4$ has no subgroup of order 6. Otherwise, if it had such a subgroup $H$ and if $\sigma$ is a 3-cycle not in $H$, then two of the cosets $H, \sigma H$ and $\sigma^2 H$ must be equal. Since $H \neq \sigma H$ we must have $H = \sigma^2 H \iff \sigma^{-1} \in H \iff \sigma \in H$ or $\sigma H = \sigma^2 H \iff \sigma \in H$. Both cases are impossible, as we chose $\sigma \notin H$.

Thus subgroup of $A_4$ can have order 1, 2, 3 or 4. Subgroups of order 3 and 2 must be cyclic, generated by elements of that order. Any subgroup of order 4 has subgroups of order 2, which must be generated by elements of the form $(12)(34)$ etc. This easily gives that $K$ is the only subgroup of $A_4$ of order 4. □

Problem 4.

Proof. The 60 rotations are: 1 identity, 12 rotations by angle $2\pi/5$, 12 rotations by angle $4\pi/5$ (for each pair of opposite faces), 20 rotations of angle $2\pi/3$, (for each vertex), and 15 rotations of angle 180, corresponding to the 15 pair of opposite edges (around the line connecting the midpoints of this edges). □

Problem 5.

Proof. Consider the set of four pairs of opposite vertices. Then $G$, the group of rigid motions of the cube, acts on this set. We thus get a homomorphism $G \to S_4$. We want to show that this is an isomorphism. Let $g$ and $h$ be distinct elements of $G$ and $v$ a vertex such that $g(v) \neq h(v)$. If $w$ is the vertex opposite to $v$, then $g(w) \neq h(w)$, so $g$ and $h$ act differently on the pair $\{v, w\}$. Thus $G \to S_4$ is injective.

On the other side, $|G| = 24$ by Ex. 10 of 1.2. □

Problem 6.

Proof. (a) $\text{id} \in A_n$ as it is the product of 0, which is an even number of permutations. If $g \in A_n$ and $g = T_1 \cdot \ldots \cdot T_{2k}$, where $T_i$ are transpositions, then $g^{-1} = T_{2k} \cdot \ldots \cdot T_1$, hence $g^{-1} \in A_n$. Finally, if $g = T_1 \cdot \ldots \cdot T_{2k}$ and $h = T'_1 \cdot \ldots \cdot T'_s$, then $gh$ is the product of $2k + 2s$ transpositions, hence $gh \in A_n$.

(b) Any permutation is the product of either an odd or an even number of transpositions. If $\sigma$ is the product of an odd number of transpositions, then $(12)\sigma$ is the product of an odd number of transpositions. If $\sigma$ is the product of an even number of transpositions, then $(12)\sigma$ is the product of an odd number of transpositions.

Thus every element of $S_n$ is either in $A_n$ or $(12)A_n$, so there are two cosets. □
Problem 7.

Proof. Left regular action simply means left multiplication. If $g$ is in the kernel, then $ga = 1$ for all $a$. This clearly implies $g = 1$, so the kernel is just $\{1\}$. \hfill \Box

Problem 8.

Proof. 14. Since $G$ is not abelian, there are elements $g, h$ with $gh \neq hg$. Assume $g \cdot a = ag$ for all $g, a \in G$ was a left group action. Then we should have $(hg)a = h(ga)$, for all $a \in G$. Take in particular $a = 1$. However, the left side is then equal to $1hg = hg$, while the right side equals $h \cdot 1g = 1gh$. This implies $hg = gh$, which is a contradiction.

15. We have $1 \cdot a = a1^{-1} = a$. If $g, h, a \in G$, then $g \cdot (h \cdot a) = (h \cdot a)g^{-1} = ah^{-1}g^{-1}$. This agrees with $gh \cdot a = a(gh)^{-1} = ah^{-1}g^{-1}$, hence we have a (left) group action. \hfill \Box

Problem 9.

Proof. $G$ acts on the cosets of $G/H$ by left multiplication, so we get a permutation representation $G \to S_{G/H} = S_n$, since $H$ has index $n$ in $G$. Let $K$ be its kernel. Then $K$ is normal in $G$, and we have an injective group homomorphism $G/K \to S_n$. Thus $|G/K| \leq |S_n| = n!$, as wanted. \hfill \Box