Ma 5a: Final Exam, December 9 at 10 am

Duration: 5 hours

The total of the following this exam is worth 130 points, but your final grade will be out of 100 points. In the case of problems with multiple parts, you may use any part in the solution of any other part even if you have not solved it.

1. (30 points) Please answer true or false for the following statements (each worth 2 points) and supply an one line brief justification to your answer (each worth 1 point). The justification will only be graded if the answer is correct.
   
   i  ______ Every group of order $2p^n$, $p$ a prime, is solvable.
   
   ii  ______ There is a solvable group with $V_4$ as one of its composition factors.
   
   iii  ______ The commutator of $S_n$ is $A_n$ for $n \geq 5$.
   
   iv  ______ Every group generated by elements of finite order is finite.
   
   v  ______ If $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ then $(m, n) = 1$.
   
   vi  ______ There exist non-abelian simple groups of arbitrary large orders. Namely, for any integer $N \geq 0$, there exists a simple group of order greater or equal to $N$ which is not abelian.
   
   vii  ______ Every Sylow $p$-subgroup of $D_{2n}$ for odd prime $p$ is cyclic.
   
   viii  ______ There exists a unique abelian group with invariant factors $3 \times 5$, $2^2 \times 3^2$, $2^5 \times 3^2 \times 5$ up to isomorphism. (i.e. There is only one isomorphism type for such abelian groups.)
   
   ix  ______ Any semidirect product $V_4 \rtimes \mathbb{Z}_5$ is abelian.
   
   x  ______ Every finite group can be written as a successive semi-direct product of its Sylow subgroups.

Solutions. (15mins)

i) T Generalized Cauchy Theorem tells us there is a subgroup of order $p^n$. It is a $p$-group so is solvable while its index 2 hence it is normal. Hence there is a derived series for the group.

ii) F Composition factors are simple but $V_4$ is not.

iii) T $1 \neq [S_n, S_n] \leq S_n$ and $[S_n, S_n] \subset A_n$ for $S_n/A_n \cong \mathbb{Z}_2$ is abelian.

iv) F Consider the infinite Dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$. Then product of two reflections is a rotation which has infinite order.

v) T Otherwise the largest order of elements won’t agree. $lcm(m, n) \neq mn$ if $(m, n) \neq 1$.

vi) T The alternating group $A_n$ are simple and nonabelian of order $n!/2$ for all $n \geq 5$.

vii) T $p$ is odd so the $p$-subgroup contains no reflection so it is a subgroup of the subgroup of rotations, which is cyclic, and hence is cyclic.

viii) F Sine these number doesn’t satisfy $d_i|d_{i+1}$ in arbitrary order. $(3 \times 5 \nmid 2^2 \times 3^2)$

ix) T $\mathbb{Z}_5 \to Aut(V_4)$ can only be trivial for $Aut(V_4)$ has order 6.

x) F Simple groups has no nontrivial proper normal subgroup and hence cannot be a semi-direct product.
2. (20 points) Classify groups of order 2015 = 5 × 13 × 31.

Solution. By Sylow’s Theorem $n_{13} = n_{31} = 1$ hence there are unique Sylow 13- and 31-subgroups. Any group $G$ of order 2015 contains a normal subgroup of order $13 \times 31$ and is a semi-direct product of $\mathbb{Z}_5 \times \phi \mathbb{Z}_{13 \times 31}$ for some homomorphism $\phi : \mathbb{Z}_5 \to Aut(\mathbb{Z}_{13 \times 31}) \cong \mathbb{Z}_{13 \times 31}^\times$. Notice that $\mathbb{Z}_{13 \times 31}^\times$ is abelian of order $12 \times 30 = 2^3 \times 3^2 \times 5$. It has a unique Sylow 5-subgroup of order 5. If $\phi$ is trivial then $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{13 \times 31} \cong \mathbb{Z}_{4 \times 13 \times 31}$. If $\phi$ is nontrivial, then the image must equal to a 5-subgroup of $Aut(\mathbb{Z}_{13 \times 31})$. By Lemma 3.13 of the Lecture, all such semi-direct product are isomorphic and isomorphic to

$$<a, b, c \mid a^{13} = b^{31} = c^5 = 1, ab = ba, ac = ca, cbc^{-1} = b^2 >.$$ 

Solution for 1209. By Sylow’s Theorem $n_{13} = n_{31} = 1$ hence there are unique Sylow 13- and 31-subgroups. Any group $G$ of order 1209 contains a normal subgroup of order $13 \times 31$ and is a semi-direct product of $\mathbb{Z}_3 \times \phi \mathbb{Z}_{13 \times 31}$ for some homomorphism $\phi : \mathbb{Z}_3 \to Aut(\mathbb{Z}_{13 \times 31}) \cong \mathbb{Z}_{13 \times 31}^\times$. Notice that $\mathbb{Z}_{13 \times 31}^\times$ is abelian of order $12 \times 30 = 2^3 \times 3^2 \times 5$. It has a unique Sylow 3-subgroup of order 9 isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. If $\phi$ is trivial then $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{13 \times 31} \cong \mathbb{Z}_{4 \times 13 \times 31}$. If $\phi$ is nontrivial, then the image must equal to a 3-subgroup of $Aut(\mathbb{Z}_{13 \times 31})$ order 3 which is contained in the unique Sylow 3-subgroup. We can see there are $4 = (9 - 1)/(3 - 1)$ many such. By Lemma 3.13 of the Lecture, this gives us at most 4 non-isomorphic semi-direct product which is nonabelian.

3. (20 points) Show that group of order 8$p^n$, $p$ an odd prime, is nonsimple. (Remark. If $p = 2$, then it is a p-group and is solvable.)

Solution. (5mins) Let $G$ be such a group. If $p > 7$ then $n_p = 1$ and the Sylow $p$-subgroup is normal and the group is nonsimple. Assume $p \leq 7$. (If $p = 2$ then this is a 2-group is solvable hence nonsimple.) If $n_p = 1$ then again the group is nonsimple. Assume $n_p > 1$ then $p = 3$ or 7. If $p = 3$ then $n_3 = 4$ and $G$ acts on $Syl_3(G)$ and there is a nontrivial homomorphism $G \to S_4$. Since $24 \mid |G|$ so either $G \cong S_4$ or the homomorphism has a nontrivial proper kernel. In both case $G$ is nonsimple. (Alternatively, if $|G| = 24$ then $n_2 \in \{1, 3\}$. If $n_2 = 3$, then there is a nontrivial homomorphism $G \to S_3$ which results in a nontrivial kernel. So there exists a nontrivial normal subgroup.) If $p = 7$ then $n_7 = 8$. Similarly there is a nontrivial homomorphism $G \to S_8$. If $n \geq 2$ then $7^2 \mid |G|$ and the homomorphism has a nontrivial kernel whose order is divisible by 7. Hence $G$ is nonsimple. We assume $|G| = 56$. Then $n_7 = 8$ implies that $G$ has $8(7 - 1) = 48$ many elements of order 7. Then there are at most 8 many elements of order a power of 2. Hence we must have $n_2 = 1$ and $G$ contains a normal Sylow 2-subgroup and is nonsimple.

4. (a) (10 points) Show that $D_{2n}$ is a semi-direct product $\mathbb{Z}_2 \times \mathbb{Z}_n$ and $D_{4n}$ is isomorphic to $\mathbb{Z}_2 \times D_{2n}$ if $n$ is odd. (Remark. The latter is not true when $n$ is even.)

(b) (10 points) Show that $Q_8$ cannot be written as a semi-direct product in any nontrivial fashion.

Solution. (20mins+5mins)

(a) $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rsr^{-1} = r^{-1} \rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$ with $\phi : \mathbb{Z}_n \to Aut(\mathbb{Z}_n)$ where $\phi(1) = \text{multiplication by } -1$. Hence $D_{4n} \cong \mathbb{Z}_{2n} \rtimes \mathbb{Z}_2$ with $\phi : \mathbb{Z}_2 \to Aut(\mathbb{Z}_{2n})$ where $\phi(1) = \text{multiplication by } -1$. On the other hand, if $n$ is odd then $\mathbb{Z}_{2n} \cong \mathbb{Z}_2 \times \mathbb{Z}_n$. We note that multiplication
by $-1$ is the trivial map on $\mathbb{Z}_2$. So we conclude that
\[ D_{4n} = < r, s | r^{2n} = s^2 = 1, srs^{-1} = r^{-1}, s > < r^2 > < r^n > \]
\[ \cong < a, b, c | a^2 = b^n = c^2 = 1, aba^{-1} = b^{-1}, abc = cb > \]
\[ \cong (\mathbb{Z}_n \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong D_{2n} \times \mathbb{Z}_2. \]

(b) Assume $Q_8$ can be written as a semi-direct product of two of nontrivial groups, then there exist subgroups $H$ and $K$ of $Q_8$ with $H \leq Q_8$ and $H \cap K = 1$. However, any two nontrivial subgroups of $Q_8$ contains $\{1, -1\}$. Hence this is impossible.

5. (Warning: The following problems are rather tricky.)

(a) (10 points) Let $G$ be a finite group. Show that if $P \in Syl_p(G)$ then the normalizer $N_G(P)$ is self-normalizing, i.e. If $H = N_G(P)$, then $N_G(H) = H$. (Remark. Indeed, all $H \leq G$ such that $H \supset N_G(P)$ is self-normalizing. That is, $N_G(P)$ is weakly abnormal.)

(b) (10 points) Assume $H \leq G$ and $\text{Aut}(H) = \text{Inn}(H)$. Show that $G = HN_G(V)$ for any $V \leq H$.

Solution. (20mins+5mins)

(a) Notice that $P$ is a Sylow $p$-subgroup of $G$ and a subgroup of $N_G(P)$, so comparing order we get it is a Sylow $p$-subgroup of $N_G(P)$ as well and clearly it is normal in $N_G(P)$. This shows it is the unique Sylow $p$-subgroup of $H = N_G(P)$. (Using the terminology from §4.4 $P$ is characteristic in $H$, i.e. All automorphism of $H$ maps $P$ to $P$.)

Notice that $N_G(H)$ acts on $H$ by conjugation. The automorphism $\phi_g$ given by conjugation by $g$ sends $P$ to $P$. Hence $N_G(H)$ normalizes $P$ and is contained in $H$. Since $H \subset N_G(H)$, they must be equal.

(b) For all $g \in G$, $\phi_g$, the conjugation by $g$, is an automorphism of $H$. Since all automorphisms are inner, there exists $h_g \in H$ such that $ghg^{-1} = h_ghhh_g^{-1}$ for all $h \in H$. In particular, $gVg^{-1} = h_ghVh_g^{-1}$ for some $h_g \in H$. This implies $h_g^{-1}g \in N_G(V)$. Hence for all $g \in G$ there exists $x \in N_G(V)$ such that $g = h_gx$. Hence $G = HN_G(V)$.

An extension\(^{(1)}\) of a group $H$ by a group $K$ is a short exact sequence $1 \rightarrow H \xrightarrow{f} G \xrightarrow{g} K \rightarrow 1$. An extension is said to be split if there exists a homomorphism $s : K \rightarrow G$ such that the composition $g \circ s = id_K$. (Equivalently, there is a subgroup $K'$ of $G$ such that $g : K' \rightarrow K$ is an isomorphism.) In such case, we say $s$ splits the exact sequence. An extension is said to be central if $f(H) \subset Z(G)$.

6. (a) (10 points) Show that if $1 \rightarrow H \xrightarrow{f} G \xrightarrow{g} K \rightarrow 1$ is split then $G \simeq H \times K$ for some $\varphi$.

(b) (10 points) Show that if $1 \rightarrow H \xrightarrow{\iota} H \times K \xrightarrow{\pi} K \rightarrow 1$ is central, then $\varphi$ is trivial. Here $\iota$ is the inclusion and $\pi$ is the natural projection.

Solution. (10mins+5mins)

(a) If the sequence is split, then there exists the homomorphism $s$ with $g \circ s = id_K$ and $H', K' \leq G$ such that $H' = f(H)$ and $g(K') = K$ with $g|_{K'}$ an isomorphism. Then we have $H' \leq G$ and $K' \simeq K$ with short exact sequence $1 \rightarrow H' \xrightarrow{f} G \rightarrow K' \xrightarrow{g} 1$ where $f$ is the inclusion and $g$ is the identity on $K'$. By

\(^{(1)}\) The set of equivalence classes of extensions is denoted by $\text{Ext}^1(K, H)$, the extension of $K$ by $H$.  

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HW8 Problem 6, this shows $G \simeq H' \rtimes K \cong H \rtimes K$ where $\varphi(k)(h) = s(k)f(h)s(k)^{-1}$ for $h \in H$, $k \in K$.

(b) We first see that with $\iota$ the inclusion and $\pi$ the natural projection, the exact sequence is split. However, by (a) and the fact that the exact sequence is central we have $\varphi(k)(h) = \iota(h) = h$ for $h \in H$, $k \in K$ for $H \iota(H)$ lies in the center. We conclude that $\varphi$ is trivial, i.e. $\varphi = id_H \in Aut(H)$. 