Action of $G$ on $S$: group hom.

This always comes with a normal subgroup of $G$.

The kernel of the homomorphism $\sigma$.

**Def 2.9** If $G$ is a group acts on a set $X$. The set

$$\left\{ g \in G \mid g \cdot x = x \quad \forall x \in X \right\}$$

is called the kernel of the action. It is a normal subgroup of $G$. If an action has trivial kernel, it is said to be faithful. If the kernel is the whole group, it's called the trivial action.

*Mon Nov. 9*

**Def 2.10** The homomorphism $\sigma : G \to S_X \ x \to g \cdot x$.

Induced by the action of $G$ on $X$, it is called the permutation representation associated to the action.

**Rmk 2.11** Let $\sigma$ be the permutation representation associated to an action of $G$ on $X$. Then, $\ker \sigma$ is the kernel of the action.

If an action is faithful, then $\sigma$ gives an injection from $G$ to $S_X$.

**Thm 2.12 (Cayley)** If $G$ is a finite group of order $n$, then $G$ is isomorphic to a subgroup of $S_n$.

*pf.* Consider the action of $G$ on $G$ by left multiplication. By HW#7 this is faithful and the permutation representation is an injection $\sigma : G \to S \cong S_n$. Then $G \cong \sigma(G) \leq S_n$. 
In general, the action by conjugation of $G$ on $G$ is not faithful. The kernel of the action is

$$Z(G) = \{ g \in G \mid zg = gz \forall g \in G \}$$

the center of $G$. ($\phi = \text{id} \in S$)

---

Example 2.14 $GL_n(V)$ acts on a $n$-dim IR-vector space by linear transformation.

---

§ 2.3 Group actions on subsets

Let $G$ be a group. We can generalize the actions of $G$ in Example 2.8 to the set of subsets or subgroups.

Example 2.15

i) If $H \subseteq G$, then $gHg^{-1} \subseteq G$. $G$ acts on the set of subgroups of $G$ by conjugation.

ii) If $H \subseteq G$, then $gH$ is a left coset of $G$.

$G$ acts on the set of left cosets $G/H$ by left multiplication.

It affords a permutation representation $\sigma : G \to S_{G/H}$.

If $[G : H] = d$, then $\sigma : G \to S_d$

iii) Similarly, $G$ acts on $\mu G$ by right multiplication $g \times \mu \to \mu \cdot g^{-1} (g, H_0) \mapsto H_0g^{-1}$

---

Remark 2.16 Consider the conjugation of $G$ on the set of subgroups.

Then $N_G(H)$ is the set that stabilizes the element $H$. 

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Previous notes

Next: Short notes

before lecture:

Modular forms

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$GL_n(\mathbb{R})$ acts on an $n$-dim IR-vector space.
Prop 2.17 Let $G$ acts on $X$. For each element $x \in X$, the set
\[ G_x = \{ g \in G \mid g \cdot x = x \} \subseteq G \]
is a subgroup of $G$. It is called the stabilizer of $x$.

pf: $1 \in G_x$ and $g, h \in G_x \Rightarrow g \cdot x = x$ and $h \cdot x = x$
\[ \Rightarrow g \cdot x = x \quad \text{and} \quad x = 1 \cdot x = h^{-1} \cdot h \cdot x = h^{-1} \cdot x \]
\[ \Rightarrow g h^{-1} \cdot x = g (h \cdot x) = g \cdot x = x \Rightarrow g h^{-1} \in G_x \]
$G_x \subseteq G$ by subgp criterion. $\Box$

Cor 2.18 Assume as above. $A \subseteq X$. Then
\[ \{ g \in G \mid g \cdot a = a \ \forall a \in A \} = \cap_{a \in A} G_a \subseteq G \]
fixer of $A$

Prop 2.19 The normalizer of a subset $A$ of $G$
defined as
\[ N_G(A) = \{ g \in G \mid g A g^{-1} = A \} \]
is a subgroup. The centralizer of $A$ defined as
\[ C_G(A) = \{ g \in G \mid g a = a g \ \forall a \in A \} \]
is a subgroup.

pf. $C_G(A) = \{ g \in G \mid g a g^{-1} = a \ \forall a \in A \} = \cap_{a \in A} G_a \subseteq G$
Under the action by conjugation of $G$ on $G$.
Consider the action of $G$ on subset of $G$ by conjugation.
Then $N_G(A) = G_a \subseteq G$. 
Remark 2.20

(i) If $A < B < G$ then $C_G(A) > C_G(B)$. $Z(G) = C_G(G)$
if $H < S$, then $C_G(H) = C_G(S)$
We don't have the same for normalizer.

(ii) If $H < G$, then $H < N_G(H)$
We don't have the same for centralizer, if $H$ is not abelian.

(iii) If $A < G$, then $C_G(A) < N_G(A)$.

Example 2.21: There exists a group that is self-normalized.

$$N_{GL_n(F)}(B) = B$$

where $B$ is the set of upper triangular matrices in $GL_n(F)$.

Remark 2.22: If $H < G$ and $N_G(H) = H$ (the smallest possible set)

Then $H$ is said to be self-normalizing. (Indeed, if $H < G$ and $V H < K < G$, $N_G(K) = K$, then $H$ is abnormal.)

Wed

§ 24 The class equation

Lemma 2.23: $G$ acts on $X$, and $x \in X$. Then $g.x = g.x \Rightarrow g^{-1} e G_x

Question: How many elements can $G$ send $x$ to?

Roughly: At most $|G|/|C_G(x)|$ many.

Question: Is $G_x$ normal?

$(g G_x g^{-1})(g x) = g x$. Indeed, $g G_x g^{-1} = G_{g x}$

Example 2.24: The group $SL_2(R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bigg| a, b, c, d \in R, \; ad - bc = 1 \right\}$

acts on the upper half plane $H = \{ z \in C \; | \; \text{Im} z > 0 \}$ of the complex plane. For $z = a + it \in H$, $z = \left[ \frac{a}{\sqrt{a^2 + t^2}} \right]_1$.
Def 2.25 Let $G$ act on $X$. The orbit of $x \in X$ under $G$ is the subset

$$O_x = \{ g \cdot x \mid g \in G \} \subset X.$$  

Can also be written as

$$G \cdot x = \{ g \cdot x \mid g \in G \}.$$  

Note that $g \cdot x = g' \cdot x \Leftrightarrow g^{-1}g' \in G_x$. We have

$$|O_x| = |G / G_x| = [G : G_x].$$

If $X = O_x$ for some/all $x \in X$, we say $G$ acts on $X$ transitively. (We say $G$ acts on $X$ simply transitively if $\forall x, y \in X$, $\exists g \in G$ s.t. $y = g \cdot x$. Then there is a bijection from $G$ to $X$ and $|X| = |G|$.)

Example 2.26 (i) $S_x$ acts on $X$ transitively.

(ii) $D_{2n}$ acts on regular $n$-gon transitively

(iii) $SL_2(\mathbb{R})$ acts on $H$ transitively but not faithfully

(\text{kernel is } \{ \pm I_2 \})

Note $[a \ b] = \begin{pmatrix} a+ib & \bar{a}+ib \end{pmatrix}$

$\Leftrightarrow a + ib = -c + di \Rightarrow a = d, b = -c$

$\Rightarrow a^2 + b^2 = 1$

$SL_2(\mathbb{R}) = \{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid \theta \in [0, 2\pi) \}.$

Prop 2.27 Orbits of an action on $X$ define an equivalence relation on $X$, i.e. $x \sim y \Leftrightarrow \exists g \in G$ s.t. $y = g \cdot x$.

(Recall we have use a similar way to show there exists a cycle for $\sigma \in S_n$, where we consider $\langle \sigma \rangle \subseteq S_n$.)

\text{Reflexivity} \quad x = \langle x \rangle \Rightarrow x \sim x

\text{Symmetry} \quad y = g \cdot x \Rightarrow x = g^{-1} y \quad \Rightarrow \quad x \sim y \Rightarrow y \sim x

\text{Transitivity} \quad y = g \cdot x, \quad z = h \cdot y \Rightarrow z = (h \cdot g) \cdot x

\Rightarrow x \sim y \text{ and } y \sim z \Rightarrow x \sim z.
Cor 2.28 If $G$ acts on $X$, then $X$ is a disjoint union of orbits. If $G$ acts on $X$ transitively then $G = G_x$ for any $x \in X$.

pf. Since $X$ is a disjoint union of equivalence classes under the relation defined in Prop 2.27, and each class is equal to $\{ g \cdot x \mid g \in G \} = G_x$ for some $x \in X$, so $X$ is a disjoint union of $G_x$'s.

If $G$ acts transitively, then there exists $x_0 \in X$ s.t. $X = G_{x_0}$. This implies there is only one orbit:

Hence, $G_x = G_{x_0}$ for all $x \in X$ and $X = G_x \forall x \in X$.

As a result, for $X$ a finite set,

$$|X| = \sum_{G_x} |G_x|$$

i.e. the sum running over all orbits in $X$.

On the other hand, we know

$$|G_x| = |G : G_x| = [G : G_x]$$

always a divisor of $|G|$.  

Prop 2.29 $|X| < \infty$ $\Rightarrow$ $|X| = \sum_{G_x} [G : G_x]$.

Consider the action of $G$ on itself by conjugation.

Then for each $x \in G$, the orbit of $G$ containing $x$ is the set of elements in $G$ conjugate to $x$:

$$\{ g \cdot x \cdot g^{-1} \mid g \in G \}.$$

This is called the conjugacy class containing $x$, denoted $Cl(x)$.

eg. In $S_n$, $Cl((12)) = \{ \sigma(12)\sigma^{-1} \mid \sigma \in S_n \} = \{ (1\sigma(12)\sigma^{-1}) \mid \sigma \in S_n \}$

$= \{ all \ 2\text{-cycles in } S_n \}$
Theorem 2.30 (Class equation) Given any finite group $G$.

The class equation of $G$ is a partition of $|G|$

$$|G| = |Z(G)| + \sum_{x \in Z(G)} \left[ G : C_G(x) \right]$$

**Proof.** Consider the action by conjugation of $G$. Then the equation

$$|G| = \sum_{x \in Z(G)} \left[ G : C_G(x) \right]$$

becomes

$$|G| = \sum_{x \in Z(G)} \left[ G : C_G(x) \right] \quad \text{as} \quad C_x = \{ g \in G \mid g x g^{-1} = x \}$$

Notice that $|C(x)| = 1 \iff G = C_G(x) \iff x \in Z(G)$.

Grouping $C(x)$ such that $x \in Z(G)$ results in

$$|G| = \sum_{x \in Z(G)} 1 + \sum_{x \notin Z(G)} \left[ G : C_G(x) \right]$$

$$= |Z(G)| + \sum_{x \notin Z(G)} \left[ G : C_G(x) \right]$$

Example 2.31. The conjugacy classes of $S_n$ are parametrized by the cycle types. A permutation $\sigma \in S_n$ has cycle type $n_1, n_2, \ldots, n_r$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ if $\sigma$ has a cycle decomposition of a product of a $n_1$-cycle, a $n_2$-cycle, and a $n_r$-cycle. (e.g., $(12)(345)(6)$ has cycle type $2 \cdot 3 \cdot 1$)

$S_4$: cycle types | representatives | size of conjugacy class

| $1+1+1+1$ | 1 | 1 |
| $1+2+1+1$ | (12) | $C_2^4 = 6$ |
| $1+3+1$ | (123) | $C_3 \cdot 2! = 8$ |
| $2+2$ | (12)(34) | $C_2^2 \cdot 2! = 3$ |
| $4$ | (1234) | $C_4 \cdot 3! = 6$ |

Class equation: $24 = 1 + 6 + 8 + 3 + 6$. 

Fri Nov 13
Trick

\[ \sigma(i_1, i_2, i_3, \ldots, i_r) \sigma^{-1} \]

- \[ \sigma(i_j) \rightarrow i_j, i_{j+1} \rightarrow \sigma(i_{j+1}) \]
- \[ \sigma(i_r) \rightarrow i_1, \sigma(i_1) \rightarrow \sigma(i_1) \]

it is \( (\sigma(i_1) \, \sigma(i_3) \ldots \, \sigma(i_r)) \)

More Properties about \( S_n \)

1) \( S_n \) is generated by all 2-cycles (transpositions)

\[ (12), (34), (123), (132), (123)(4), (124)(3) \]

\[ (12)(34), (13)(24) \]

\[ (12)(34), (123)(4) \]

2) Permutations admits a cycle decompositions

3) \( r \) cycles have order \( r \)

4) \( \alpha, \beta \in S_n \) Then \( \alpha, \beta \) are conjugate \( \iff \) \( \alpha, \beta \) has same cycle type

\[ \alpha = (a_1, a_2, \ldots, a_r) \cdot (b_1, b_2, \ldots, b_k) \cdot \ldots \cdot (c_1, c_2, \ldots, c_l) \]

\[ \Rightarrow \sigma \alpha \sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_r)) \cdot (\sigma(b_1), \sigma(b_2), \ldots, \sigma(b_k)) \cdot \ldots \cdot (\sigma(c_1), \sigma(c_2), \ldots, \sigma(c_l)) \]

if \( \beta = (a'_1, a'_2, \ldots, a'_r) \cdot (b'_1, b'_2, \ldots, b'_k) \cdot \ldots \cdot (c'_1, c'_2, \ldots, c'_l) \)

Take \( \sigma \) such that \( a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_k \rightarrow c_1, c_2, \ldots, c_l \)

then \( \beta = \sigma \alpha \sigma^{-1} \)

5) \( S_n = \langle (12), (1234 \cdot n) \rangle = \langle (12), (13), \ldots, (n-1, n) \rangle = \langle (12), (13), \ldots, (1n) \rangle \)
Application of Class Eq.

Theorem 2.3.2 (Cauchy's Theorem)

Assume $G$ is a finite group and $p$ is a prime divisor of $|G|$. Then $\exists a \in G$ such that $o(a)=p$.

(NB: We have shown this for $G$ abelian.)

pf. As before, we do induction on $|G|$. If $|G|=1$, then this is trivially true. Assume true for all groups of order $<|G|$. If $\exists H \leq G$ s.t. $p|\text{Ind(H)}$, then by induction hyp. $\exists a \in H$ s.t. $o(a)=p$ and $a \in G$. Assume not.

Consider the subgroup $C_G(a)$ for all $a \in \mathbb{Z}(G)$. Then we have $p \nmid |C_G(a)|$ by assumption. This implies $[G : C_G(a)]$ is divisible by $p$.

The class equation

\[
|G| = |\mathbb{Z}(G)| + \sum_{a \in \mathbb{Z}(G)} [G : C_G(a)].
\]

then implies $p \mid |\mathbb{Z}(G)|$. By assumption $\mathbb{Z}(G)$ needs to be $G$.

However, then $G$ is abelian. Since the statement is true for all abelian groups, we get $\exists a \in G$ s.t. $o(a)=p$.

Example 2.3.3. Assume $H \leq S_5$ and $|H|=15$. Then by Cauchy's Thm. $\exists a, b \in S_5$ $o(a)=3$, $o(b)=5$. Apply the cycle decompt. of $o, b$, then we deduce $a$ is a 3-cycle and $b$ is a 5-cycle. Since $15|ka\cdot b\rangle$, such $H$ is generated by a 3-cycle and a 5-cycle.

NB In $S_5$, $(123)(456)$ has order 3.
Prop 2.34. If \( G \) is a group of order \( p^2 \), \( p \) prime, then 
\( G \) is abelian.

pf. We have seen before that \( |G| = p^2 \Rightarrow G \) abelian or \( \mathbb{Z}(G) = 1 \).

By class equation,
\[
|G| = |\mathbb{Z}(G)| + \sum [G : C_G(x)]
\]
\( x \in \mathbb{Z}(G) \)

pf. \( \mathbb{Z}(G) \) cyclic

Since for \( x \notin \mathbb{Z}(G) \), \( [G : C_G(x)] \neq 1 \) and is a divisor of \( \frac{|G|}{|\mathbb{Z}(G)|} \) if \( \mathbb{Z}(G) \neq 1 \).

Hence \( \frac{|G|}{|\mathbb{Z}(G)|} \) is a multiple of \( p \), so it is a multiple of \( p^2 \). Comparing both sides,

we get \( p \mid |\mathbb{Z}(G)| \). Since \( 1 \in \mathbb{Z}(G) \Rightarrow 1 \leq |\mathbb{Z}(G)| \),

we see \( \mathbb{Z}(G) \neq \{1\} \) and hence \( G \) is abelian.

In general we can show the following.

Cor 2.35. If \( G \) is a \( p \)-group, then \( \mathbb{Z}(G) \neq 1 \).

pf. Similar to the proof above.

§2.5 Symmetric groups and alternating groups.

By Cayley's Theorem, all finite groups are isomorphic to a subgroup of \( S_n \). Let us investigate \( S_n \) a little more.

Observation: \( S_n = \{ \text{permutations} \} \) can be generated by

swapping two numbers, i.e. \( S_n = \langle (ab) \mid 1 \leq a < b \leq n \rangle \).

It's then easy to see that \( S_n = \langle (12), (23), \ldots, (n-1\ n) \rangle \)

(\text{generated by swapping adjacent numbers}).

Facts: i) \((ab) = (1b)(1a)\), \((abc) = (1c)(1b)(1a)\)

ii) \((ab) = (1b)(1a)(1b)\), \((abc) = (1c)(1b)(1a)(1c)\)

Hence \( S_n = \langle (1\ i) \mid 1 \leq i \leq n \text{ integer} \rangle \).
In homework, we introduced subset $A_n$ of $S_n$ which forms an index 2 normal subgroup. As $S_n$ is not commutative (indeed $\mathbb{Z}(S_n) = 1$), there is no unique way to write elements in $S_n$ as product of 2-cycles. Here we provide a way to make even and odd well-defined.

NB: $(a_1 a_2 \cdots a_r) = (a_1 a_r)(a_1 a_{r-1}) \cdots (a_1 a_3)(a_1 a_2)

-r$-cycle $r+1$ many transpositions

To detect the parity in $S_n$:

$$
\Delta = (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_n)
\cdot (x_2 - x_3) \cdots (x_2 - x_n)
\cdots (x_n - x_n)
= \prod_{1 \leq i < j \leq n} (x_i - x_j)
$$

Set $\sigma \in S_n$

$$
(\sigma \Delta)(x_1, \ldots, x_n) := \Delta (x_{\sigma(1)}, \ldots, x_{\sigma(n)})
= \prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)})
$$

Then

$$
|\sigma \Delta| = |\Delta| \quad \text{i.e.} \quad \sigma \Delta = \pm \Delta \quad \text{sign dep. on } \sigma.
$$

Def: sign function $\varepsilon : S_n \to \{-1, 1\}$ by $\varepsilon(\sigma) = \frac{\sigma \Delta}{\Delta}$

Prop 2.36 $\varepsilon$ is a homomorphism and $\ker \varepsilon = A_n$

pf.

$$
1 \cdot \Delta = (\varepsilon(\sigma) \Delta) = \varepsilon(\sigma) 1 \Delta = \varepsilon(\sigma) \varepsilon(\tau) \Delta
$$

$(ab) \sim (12) \Rightarrow \varepsilon(ab) = \varepsilon(12) = -1 \Rightarrow A_n < \ker \varepsilon \leq S_n$
Def 2.37. $A_n$ is called the alternating group of degree $n$, $n \geq 4$

A permutation is even if it is in $A_n$ and odd otherwise.

\[ \sigma \in S_n \Rightarrow \sigma = \begin{cases} 
\text{even} & \text{if } \varepsilon(\sigma) = 1 \\
\text{odd} & \text{if } \varepsilon(\sigma) = -1 
\end{cases} \]

Remark 2.38

(i) $A_4$ is the rotation group of tetrahedron.
and $A_4$ is solvable.

(ii) $A_4 \cong SL_2(\mathbb{Z}/2\mathbb{Z})/\{\pm I\}$ order 12.

(iii) $A_5 \cong SL_2(\mathbb{F}_4)$, where $\mathbb{F}_4$ is a field of order 4.

(iv) In general, $PSL_2(\mathbb{F}_q)$ is simple for $q > 3$ odd.

Prop 2.39. $A_5$ is a simple group of order 60.

In general,

\[ |A_n| = |Z(A_n)| + \sum_{\sigma \neq e} [A_5 : C_5(\sigma)] \]

where $C_5(\sigma)$ is the centralizer of $\sigma$ in $A_5$.

\[ |A_5| = 60 \]

Compare $[S_5 : C_5(\sigma)]$ and $[A_5 : C_5(\sigma)]$ for $\sigma \in S_5$.

**NB:** $C_{A_5}(\sigma) = C_5(\alpha) \cap A_5$.

By 2nd Isom Thm,

\[ [A_5 : C_5(\sigma)] = [A_5 : C_3(\sigma)] \cdot [S_5 : C_5(\sigma)] \]

Since $\sigma$ is even,

\[ \begin{aligned}
&\sigma : \text{even} \\
&\sigma : \text{odd} \\
&1, (123), (12)(34) \quad [S_5 : C_5(\sigma)] / 2 \\
&(12345), (123) \quad (134) \quad (12345) \quad (21345) \quad (21345)
\end{aligned} \]

60 = $1 + C_5^2 C_2^3 + C_5^2 C_2^3 / 2 + C_5^2 - 4 \cdot \frac{1}{2}$ + \[ \begin{array}{cccc}
1 & 20 & 15 & 12 \\
12 & & & \end{array} \]

If $H \leq A_5$, then $|H| | 60$ and $|H| - 1$ sum of pair of $5, 20, 15, 12, 12$.
Prop. 2.40.\ An is simple for all $n \geq 5$

If $\text{Induction on } n$. Assume $\text{An}$ is simple for $5 \leq m < n$.
(We know $\text{An}$ is simple for $m = 5$) and $n \geq 6$.
Assume $N \not\subset \text{An}$, and $N \neq \{1\}$ or $\text{An}$.
Consider subgroups $H_i = (A_{n-1})$, where $A_{n-1}$ acts on $X_n$ by permutations. Then $H_i \cong A_{n-1}$ (even perm on $X_{n-1}$).
Since $H_i; \bar{\sigma}$ are simple, $H_i \cap N = \{1\}$ or $H_i \not\subset N$.

1) If $H_i \cap N = H_i$, then $H_i < N \not\subset \text{An}$.
However $N$ is normal $\Rightarrow N \supseteq \bigcup_{\bar{\sigma} \in \text{An}} \bar{\sigma} H_i \bar{\sigma}^{-1} = \bigcup_{\bar{\sigma} \in \text{An}} H_i$
For $n \geq 3$, $A_n$ acts on $X_n$ transitively. $(12 \ast)$
Hence $N \supseteq \bigcup_{i=1}^n H_i$. However $\forall (ab)(cd) \in A_n$
$(ab)(cd) \in H_i$ for some $i$ if $n \geq 5$.
$A_n = \langle (ab)(cd) \mid (ab)(cd) \in S_n \rangle$
$= \langle H_i; \ i = 1, 2, \ldots, n \rangle \subset N$

$\therefore N = A_n$.

2) If $H_i \cap N = \{1\}$, then $N$ fixes no $i \in \{1, 2, \ldots, n\}$.
This forces $\forall \bar{\sigma} \not= \bar{\iota} \in N, \bar{\sigma}(i) \neq \bar{\iota}(i)$ $\forall i$ (otherwise $\bar{\sigma}^{-1} \bar{\iota} \in H_i \cap N = \{1\}$, thus $\bar{\sigma} = \bar{\iota}$ $\ast$ $\ast$).
But if $\sigma = (a_1 a_2 a_3 \ldots)$ is in $N$.
Take $b c d \in X_n$ with $\{a_1, a_2, a_3 \}$ distinct.
Then $\bar{\sigma} (a_1 a_2 a_3 \ldots) \in N$.
With $\bar{\sigma}(a_1) = a_2 = \tau(a_1)$ and $\bar{\tau} \not\subset \sigma$.
If $\sigma = (a_1 a_2)(b_1 b_2) \ldots (c_1 c_2)$ is in $N$.
Take $\chi = (a_1 a_2)(b_1 c_1) \in A_n$, then $\tau = \bar{\sigma} \bar{\chi}^{-1} \in N$.
And $\tau(a_1) = a_2 = \sigma(a_1)$ with $\tau \not\subset \sigma$.
Similarly $\bar{\sigma} = (ab)(cd)$ does not sit in $N$ ($\bar{\sigma} = (ab)(cd)$).
Hence $N \subset \{1\}$. $\ast$ $\ast$ $\ast$