1. (10 points) Section 5.2
Find the power series solutions, and the recurrence relation of the differential equation about \( x = 0 \),
\[ y'' - y = 0. \] (1)

solution

Let
\[ y = \sum_{n=0}^{\infty} a_n x^n. \]

Then compute directly to have
\[ y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n, \]
where in the last step we changed variable \( n \to n + 2 \).

By the equation for \( y \) we derive
\[ 0 = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - a_n]x^n. \]

By this we have the recurrence relation
\[ a_{n+2} = \frac{1}{(n+2)(n+1)} a_n. \]

Consequently we have for even \( n \),
\[ a_n = \frac{1}{n!} a_0, \]
for odd \( n \),
\[ a_n = \frac{1}{n!} a_1. \]
2. (10 points) Section 5.2

Find the recurrence relation of the differential equation about \( x = 0 \),

\[
2y'' + xy' + 3y = 0. \tag{2}
\]

**solution**

Let

\[
y = \sum_{n=0}^{\infty} a_n x^n.
\]

Then compute directly to have

\[
y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n,
\]

where in the last step we changed variable \( n \rightarrow n+2 \).

Similarly we have

\[
xy' = \sum_{n=0}^{\infty} a_n nx^n.
\]

Plug these into the equation to find

\[
\sum_{n=0}^{\infty} x^n[2a_{n+2}(n+2)(n+1) + a_n n + 3a_n] = 0.
\]

Then the recurrence relation is

\[
2a_{n+2}(n+2)(n+1) + a_n n + 3a_n = 0
\]

or

\[
a_{n+2} = -a_n \frac{n + 3}{2(n+2)(n+1)}.
\]
3. (10 points) Section 5.4
Determine the singular point of the Euler’s equation, and find two general solutions to the differential equation

\[ x^2 y'' - 3xy' + 4y = 0. \]  \hspace{1cm} (3)

solution The singularity is at \( x = 0 \).
To find two solutions we have \( y = x^r \), with \( r \) satisfying the equation

\[ r(r - 1) - 3r + 4 = 0, \text{ or } r^2 - 4r + 4 = 0. \]

The root is \( r = 2 \) with multiplicity 2.
Thus the differential equation has two solutions

\[ y = x^2, \; x^2 \ln x. \]
4. (10 points) Section 5.4

Find the solution satisfying the given initial condition

\[ x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1. \] \hspace{1cm} (4)

To find two solutions we have \( y = x^r \), with \( r \) satisfying the equation

\[ r(r - 1) + 3r + 5 = 0, \quad \text{or} \quad r^2 - 2r + 5 = 0. \]

The roots are \( r_1 = 1 + 2i, \quad r_2 = 1 - 2i. \)

Hence the differential equation has two solutions

\[ y = x^{1+2i}, \quad x^{1-2i}, \]

or

\[ y = x\cos(lnx), \quad x\sin(lnx). \]

The general solution is of the form

\[ y = c_1x\cos(lnx) + c_2x\sin(lnx). \]

To fix \( c_1, \ c_2 \) we have the initial condition to have \( c_1 = 1, \) and \( c_1 + c_2 = -1. \)

Thus \( c_1 = 1, \ c_2 = -2. \)

The solution is

\[ y = x\cos(lnx) - 2x\sin(lnx). \]
5. (10 points) Section 5.5
Find the indicial equation, the recurrence relation, and the roots of the indicial equation

\[ x^2 y'' + xy' + (x^2 - \frac{1}{9})y = 0. \]  

(5)

solution

The indicial equation is

\[ r(r - 1) + r - \frac{1}{9} = 0. \]

It has two roots, \( r_1 = \frac{1}{3}, \quad r_2 = -\frac{1}{3} \).

The two solutions take the form

\[ y_1 = x^{\frac{1}{3}} \sum_{n=0}^{\infty} a_n x^n, \]

and

\[ y_2 = x^{-\frac{1}{3}} \sum_{n=0}^{\infty} b_n x^n. \]

Here the recurrence relation for \( a_n \) is

\[ a_n[(n + \frac{1}{3})(n - \frac{2}{3}) + (n + \frac{1}{3}) - \frac{1}{9}] = -a_{n-2}, \]

or

\[ a_n = -\frac{1}{(n + \frac{1}{3})(n - \frac{2}{3}) + (n + \frac{1}{3}) - \frac{1}{9}}a_{n-2}. \]

Similarly for \( b_n \) we have

\[ b_n = -\frac{1}{(n - \frac{1}{3})(n - \frac{4}{3}) + (n - \frac{1}{3}) - \frac{1}{9}}b_{n-2}. \]
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