1. (10 points) Draw a direction field for the following equation

\[ y' - 2y = 3e^t. \]

Then determine the behavior of \( y \) as \( t \to \infty \), and find the general solution of the given equation.

The direction field is

![Direction Field](image.png)

solution It is a linear first order ODE. The integrating factor is \( e^{-2t} \). Multiply both sides by \( e^{-2t} \) to obtain

\[ e^{-2t}y' - 2e^{-2t}y = 3e^{-t}. \]

Hence

\[ \frac{d}{dt}[e^{-2t}y] = 3e^{-t}. \]

Integrate both sides to obtain

\[ e^{-2t}y = -3e^{-t} + C. \]
Multiply both sides by $e^{2t}$ to obtain

$$y = -3e^t + Ce^{2t}.$$

As $t \to \infty$, then $y(t)$ will grow as $Ce^{2t}$ if $C \neq 0$. If $C = 0$, then $y$ will grow as $-3e^t$.

2. (10 points) Find the value of $y_0$ for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t.$$

remains finite as $t \to \infty$.

**solution** The integrating factor is $e^{-t}$. Multiply both sides by $e^{-t}$ and then integrate to find

$$e^{-t}y = -e^{-t} - \frac{3}{2}e^{-t}[\cos t + \sin t] + C$$

here we used the integration

$$\int e^{-t}\sin t \, dt = -\frac{1}{2}e^{-t}[\cos t + \sin t] + C.$$

This directly implies that

$$y = -1 - \frac{3}{2}[\cos t + \sin t] + Ce^t.$$

To make $y$ bounded as $t \to \infty$, one must have $C = 0$. This is equivalent to impose the initial condition

$$y(0) = \frac{5}{2}.$$

3. (10 points) Solve the initial value problem

$$y' = \frac{2\cos 2x}{3 + 2y},$$

$$y(0) = 0.$$

Then determine where the solution attains its minimum value.

**solution** Rewrite the equation as

$$(3 + 2y)dy = 2\cos 2xdx.$$

Integrate both sides to have

$$y^2 + 3y = \sin 2x + C.$$

To fix $C$, use the condition $y(0) = 0$ to have

$$y^2 + 3y = \sin 2x.$$

(1)
Compute directly to have
\[ y = \frac{3}{2} \pm \sqrt{\frac{9}{4} + \sin 2x}. \]
Use the initial condition again to discard the \(-\) sign to have
\[ y = \frac{3}{2} - \sqrt{\frac{9}{4} + \sin 2x}. \]  
(2)

It is easy to see the minimum value of \( y \) is achieved at the maximum value of \( \sin 2x \), which are at
\[ 2x = 2k\pi + \frac{\pi}{2}, \quad k \in \mathbb{Z}. \]

4. (10 points) Solve the initial value problem
\[
y' = \frac{3x^2}{3y^2 - 4}, \quad y(1) = 0, \]
and determine the interval in which the solution is valid (justification is needed!), and plot the solution.

solution
Rewrite the equation as
\[(3y^2 - 4)dy = 3x^2dx.\]
Integrate both sides to have
\[ y^3 - 4y = x^3 + C. \]
Use the initial condition to determine \( C = -1 \). Hence
\[ y^3 - 4y = x^3 - 1. \]
The solution is valid in the interval
\[ 1 - \sqrt{\frac{4}{3}(4 - \frac{4}{3})} \leq x^3 \leq 1 + \sqrt{\frac{4}{3}(4 - \frac{4}{3})} \]
or
\[-(\frac{16}{\sqrt{3}} - 1)^{1/3} \leq x \leq (\frac{16}{\sqrt{3}} + 1)^{1/3}. \]

since the right hand side of (3) become singular at \( y^2 = \frac{4}{3} \) or \( y = \pm \sqrt{\frac{4}{3}} \), the edges of the \( x \)-interval above correspond to these values of \( y \), and \( x = 0 \) is included in this interval.

Plot
5. (20 points) A body of mass $m$ is projected vertically upward with an initial velocity $v_0$ in a medium offering a resistance $k|v|$, where $k$ is a constant. Assume that the gravitational attraction of the earth is constant.

(a) (7 points) Find the velocity $v(t)$ of the body at any time.

(b) (7 points) Use the result of Item [a] to calculate the limit of $v(t)$ as $k \to 0$, as the resistance approaches zero. Does this result agree with the velocity of mass $m$ projected upward with an initial velocity $v_0$ in a vacuum?

(c) (6 points) Use the result of Item [a] to calculate the limit of $v(t)$ as $m \to 0$, that is, as mass approaches zero.

\textbf{solution}

We start with part (a):

The problem has two parts: one part is about the object moving upward until its speed becomes zero, then after that, the second part, is about falling downward.

For the upward, we have

$$m \frac{dV}{dt} = -kV - gm.$$ 

Solve this equation to have

$$V(t) = -\frac{mg}{k} + e^{-\frac{k}{m}t}C.$$ 

To fix $C$, by the initial condition we have

$$C = V(0) + \frac{mg}{k}.$$ 

Use $T$ to denote the moment $V = 0$, then

$$T = -ln\left(\frac{mg}{KV(0) + mg}\right).$$
After $T$, the object will fall downward. The equation becomes

\[ \frac{m}{dt} = -kV + gm. \]

After similar procedure as above we have

\[ V(t + T) = \frac{mg}{k} (1 - e^{-\frac{k}{m}t}). \]

For the b part, we have, for the upward direction,

\[ V = \frac{gm}{k} (e^{-\frac{k}{m}t} - 1) + V(0)e^{-\frac{k}{m}t}. \]

As $k \to 0$ one has

\[ V \to -gt + V(0) \]

which is the solution to the ODE

\[ \frac{m}{dt} = -gm. \]

Similarly for the downward direction.

For the c part, for the upward direction, one has, for any $t > 0$, as $m \to 0$,

\[ V(t) \to 0. \]

Similarly for the downward direction.

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