Problem 1-10 pts (Problem 5.9) Suppose that $f$ is continuous at $x$. Show that
$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{x}^{x+\delta} f(\tilde{x})d\tilde{x}.$$ 

Solution. Fix $\epsilon > 0$. Choose $\delta > 0$ such that $|\tilde{x} - x| \leq \delta \Rightarrow |f(\tilde{x}) - f(x)| \leq \epsilon$.

Then for all $|\delta x| \leq \delta$ we have
$$\left| f(x) - \frac{1}{\delta x} \int_{x}^{x+\delta} f(\tilde{x})d\tilde{x}\right| = \frac{1}{|\delta x|} \int_{x}^{x+\delta} |f(\tilde{x}) - f(x)|d\tilde{x} \leq \frac{1}{|\delta x|} (x+\delta x - x)\epsilon = \epsilon$$
As $\epsilon > 0$ was arbitrary this proves the claim.

Problem 1-10 pts (Problem 8.7) Show that for $k \neq 0$ the solution of the IVP
$$\dot{x} = kx - x^2, \quad x(0) = x_0$$
is
$$x(t) = \frac{ke^{kt}x_0}{x_0(e^{kt} - 1) + k}$$

Using this explicit solution describe the behavior of $x(t)$ as $t \to \infty$ for $k < 0$ and $k > 0$.

Solution. Taking the partial $\frac{\partial}{\partial x} kx - x^2$, we see that $\dot{x}$ is continuously differentiable everywhere. Therefore, by Theorem 6.2, any solution to the IVP is locally unique. For the given $x$, we check:
$$\frac{d}{dt} x(t) = k \frac{ke^{kt}x_0}{x_0(e^{kt} - 1) + k} - (ke^{kt}x_0)(x_0(e^{kt} - 1) + k)^{-2}(kx_0e^{kt})$$
$$= kx(t) - \frac{(ke^{kt}x_0)^2}{(x_0(e^{kt} - 1) + k)^2}$$
$$= kx(t) - x(t)^2$$
And $x(0) = \frac{kx_0}{k} = x_0$, so we see that the given $x$ satisfies the IVP.

Notice that $x(t)$ might not be defined everywhere for certain $k$ and $x_0$ because the denominator can be zero. However, it is well-behaved at infinity, so we can take the limits required. First note that if $x_0 = 0$, then it doesn’t matter what $k$ is, and $x(t) = 0$, and if $x_0 = k$, then $x(t)$ is the constant function at $k$. We shall assume neither of these cases occurs.

If $k > 0$, then
$$\lim_{t \to \infty} \frac{ke^{kt}x_0}{x_0(e^{kt} - 1) + k} = \lim_{t \to \infty} \frac{k^2 x_0 e^{kt}}{kx_0 e^{kt}}$$
$$= \lim_{t \to \infty} \frac{k^2 x_0}{k x_0}$$
$$= k$$
Where we have used L’Hopital’s rule in the first equality.

If $k < 0$, then $\lim_{t \to \infty} \frac{ke^{kt}x_0}{x_0(e^{kt} - 1) + k} = \frac{0}{k-x_0}$, where we have just used the fact that a limit of a fraction is the fraction of the limits, provided the denominator isn’t zero, and our assumption that $k \neq x_0$. 

□
Problem 2-10 pts (Problem 8.11) Assuming that \( f(x) \) is continuously differentiable, show that if the solution of
\[
\dot{x} = f(x), \quad x(0) = x_0
\]blows up to \( x = +\infty \) in finite time, then
\[
\int_{x_0}^{\infty} \frac{dx}{f(x)} < \infty
\]

**Solution.** Using pages 64–65, we know that \( \frac{dx}{dt} = f(x) \) means that
\[
\int_{x_0}^{x(t)} \frac{1}{f(x)} dx = \int_{0}^{t} 1 dt
\]
We assume that the “blow up” occurs at finite time \( t = T \). If we take the limit of both sides of the equation above at \( t \to T \), we get:
\[
\lim_{t \to T} \int_{x_0}^{x(t)} \frac{1}{f(x)} dx = \int_{x_0}^{\infty} \frac{1}{f(x)} dx
\]
on the one hand, and
\[
\lim_{t \to T} \int_{0}^{t} 1 dt = T
\]
On the other, so we see that:
\[
\int_{x_0}^{\infty} \frac{1}{f(x)} dx = T < \infty
\]
□

Problem 3 10 points Draw the phase diagram and label the stationary points as stable or unstable.

**Solution.** Consider the system \( \dot{x} = (1 + x)(2 - x) \sin(x) \) and let \( f(x) = (1 + x)(2 - x) \sin(x) \). Since the stationary points of the system are precisely the roots of \( f \), the set of stationary points is \( \{-1, 2, n\pi : n \in \mathbb{Z}\} \).

\[
f'(x) = (2 - x) \sin(x) - (1 + x) \sin(x) + (1 + x)(2 - x) \cos(x) = (1 - 2x) \sin(x) + (1 + x)(2 - x) \cos(x).
\]
\[
f'(-1) = (1 + 2) \sin(-1) + (1 - 1)(2 + 1) \cos(-1) = 3 \sin(-1) < 0, \quad f'(2) = (1 - 4) \sin(2) + (1 + 2)(2 - 2) \cos(2) = -3 \sin(2) < 0 \quad \text{and for } n \in \mathbb{Z}, \ f'(n\pi) = (1 - 2n\pi) \sin(n\pi) + (1 + n\pi)(2 - n\pi) \cos(n\pi) = (1 + n\pi)(2 - n\pi)(-1)^n. \]
When \( n < 0, \ 1 + n\pi < 0 \) and \( 2 - n\pi > 0 \Rightarrow (1 + n\pi)(2 - n\pi) < 0 \). When \( n > 0, \ 1 + n\pi > 0 \) and \( 2 - n\pi < 0 \Rightarrow (1 + n\pi)(2 - n\pi) < 0 \). When \( n = 0, \ (1 + 0)(2 - 0) = 2 \). So \( f'(0) = 2 \cos(0) = 2 > 0 \) and for \( n \in \mathbb{Z} - \{0\} \), \( f'(n\pi) \) has the opposite sign as \((-1)^n\), i.e. \( f'(n\pi) \) is negative when \( n \) is even and positive when \( n \) is odd. Thus the set of stable stationary points is \( \{-1, 2, 2n\pi : n \in \mathbb{Z} - \{0\} \} \) while the set of unstable stationary points is \( \{0, (2n + 1)\pi : n \in \mathbb{Z}\} \).

The set of open intervals whose endpoints are stationary points is \( \{(-\pi, -1), (-1, 0), (0, 2), (2, \pi), (n\pi, (n + 1)\pi) : n \in \mathbb{Z} - \{-1, 0\} \} \). Since \( f \neq 0 \) on each of these intervals and \( f \) is continuous, the Intermediate Value Theorem implies that \( f \) has constant sign on each of these intervals. \( f(-3/2) = (-1/2)(7/2) \sin(-3/2) > 0 \Rightarrow f > 0 \) on \((-\pi, -1), \ f(-1/2) = (1/2)(5/2) \sin(-1/2) < 0 \Rightarrow f < 0 \) on \((-1, 0), \ f(1) = (2)(1) \sin(1) > 0 \Rightarrow f > 0 \) on \((0, 2), \ f(3) = (4)(-1) \sin(3) < 0 \Rightarrow f < 0 \) on \((2, \pi)\).

For \( n > 0 \) or \( n < -1, \ f((n + 1/2)\pi) = (1 + (n + 1/2)\pi)(2 - (n + 1/2)\pi) \sin((n + 1/2)\pi) \) has the opposite sign to \( \sin((n + 1/2)\pi) \), which in turn is positive for \( n \) even and negative for \( n \) odd. So \( f > 0 \) on \((n\pi, (n + 1)\pi) \) for \( n \) odd and \( f < 0 \) on \((n\pi, (n + 1)\pi) \) for \( n \) even in either case, provided that \( n \in \mathbb{Z} - \{-1, 0\} \).

Phase diagram omitted. □

Problem 4 10 points Draw the phase diagram and label the stationary points as stable or unstable.
Problem 5 10 points Show that for autonomous scalar equations, if a stationary point is attracting, then it must be stable.

Solution. Consider the system \( \dot{x} = x^2 - x^4 \) and let \( f(x) = x^2 - x^4 = x^2(1 - x^2) = x^2(1 - x)(1 + x) \). Since the stationary points of the system are precisely the roots of \( f \), the set of stationary points is \( \{-1, 0, 1\} \).

\[ f'(x) = 2x - 4x^3 = 2x(1 - 2x), \quad f'(-1) = -2(1 - 2) = 2 > 0, \quad f'(0) = 0(1 - 0) = 0 \quad \text{and} \quad f'(1) = 2(1 - 2) = -2 < 0. \] It follows that \(-1\) is unstable and \(1\) is stable; the stability of \(0\) will be determined later.

The set of open intervals whose endpoints are stationary points is \( \{(-\infty, -1), (-1, 0), (0, 1), (1, \infty)\} \). Since \( f \neq 0 \) on each of these intervals and \( f \) is continuous, the Intermediate Value Theorem implies that \( f \) has constant sign on each of these intervals. \( f(-2) = 4(1 - 4) = -12 < 0 \Rightarrow f < 0 \) on \( (-\infty, -1) \). \( f(-1/2) = 1/4(1 - 1/4) = 3/16 > 0 \Rightarrow f > 0 \) on \( (-1, 0) \). Because \( f \) is clearly even, \( f > 0 \) on \((0, 1)\) and \( f < 0 \) on \((1, \infty)\). We also see that for any trajectory \( x(t) \) starting at some \( \delta \in (0, 1) \), it is not true that \( |x(t)| < 1/2 \) for all \( t \geq 0 \) because \( x(t) \rightarrow 1 \) as \( t \rightarrow \infty \). It follows that \(0\) is unstable.

Phase diagram omitted.

Problem 6-10 pts Does the IVP \( \dot{x} = \frac{\sqrt{|x|}}{1 + x^2} \) with \( x(0) = 0 \) have a unique solution on \( \mathbb{R} \)?
Solution. The answer is no and so we will construct two distinct solutions. We trivially see that \( x_1(t) = 0 \) is a solution so we will proceed to construct a nonzero solution. Consider the function \( g(w) : (0, \infty) \to (0, \infty) \) given by

\[
g(w) = 2\sqrt{w} + \frac{2}{5}w^{5/2}.
\]

It is trivial to verify that \( g \) is strictly increasing on \((0, \infty)\) and tends to infinity as \( w \) does. Therefore, \( g \) is a bijection and hence invertible. We can therefore define a function \( w(t) \) so that \( g(w(t)) = t \) for all \( t > 0 \). Let us define the function \( x_2(t) \) by setting \( x_2(t) = 0 \) for \( t \leq 0 \) and equal to \( w(t) \) otherwise. From our definition and the Inverse Function Theorem, it is easy to see that \( x_2(t) \) is differentiable on \((-\infty, 0)\) and has the proper initial condition.

If we implicitly differentiate the relation \( t = 2\sqrt{w} + \frac{2}{5}w^{5/2} \) we find that

\[
1 = \left( w_{-1/2} + w_{3/2} \right) \dot{w}(t).
\]

Consequently, we have \( \dot{x}_2(t) = \sqrt{|x_2|} + x_2^2 \) for \( t > 0 \). All that remains is to show that \( x_2(t) \) satisfies the differential equation at \( t = 0 \), that is, we need to show that \( x_2 \) is differentiable at 0 and has derivative 0 there (since \( x_2(0) = 0 \)). Now, notice that

\[
\lim_{t \to 0^+} \dot{x}_2(t) = 0.
\]

The Mean Value Theorem then implies that

\[
\lim_{t \to 0^+} \frac{x_2(t) - x_2(0)}{t - 0} = 0
\]

while trivially we have

\[
\lim_{t \to 0^-} \frac{x_2(t) - x_2(0)}{t - 0} = 0.
\]

It follows that \( x_2(t) \) is differentiable at 0 with derivative equal to 0 as desired. □

Problem 7-10 pts

Problem 9.7 Suppose that \( x(t) \) satisfies the differential inequality

\[
\dot{x} \leq ax.
\]

Show that

\[
\frac{d}{dt} [e^{-at}x] \leq 0
\]

and deduce that

\[
t > s \implies x(t) \leq x(s)e^{a(t-s)}.
\]

Solution. We begin by differentiating using the product rule. We have

\[
\frac{d}{dt} [e^{-at}x] = e^{-at} \dot{x} - axe^{-at} \leq e^{-at}ax - axe^{-at} = 0
\]

proving the first claim. Suppose now for contradiction that there exists some pair \((s, t)\) with \( t > s \) such that \( x(t) > x(s)e^{a(t-s)} \). Then

\[
x(t)e^{-at} > x(s)e^{-as}
\]

and so by the Mean Value Theorem there exists \( r \in (s, t) \) with

\[
\frac{d}{dy} [e^{-ay}x(y)] \bigg|_{y=r} > 0
\]

contradicting what we showed in our first claim. Thus no such pair \((s, t)\) exists and the claim is proved. □

Problem 8-10 pts
Problem 6.3 This exercise gives a proof of uniqueness of solutions of
\[ \dot{x} = f(x, t), \quad x(t_0) = x_0 \]
under the assumption
\[ |f(x, t) - f(y, t)| \leq L|x - y|. \]
Suppose \( x(t) \) and \( y(t) \) are two solutions and let \( z(t) = x(t) - y(t) \) so
\[ \frac{d}{dt} |z|^2 = 2z[\dot{f}(x(t), t) - f(y(t), t)] \leq 2L|z|^2. \]
Use the previous exercise to complete the proof.

Solution. Observe that
\[ \frac{d}{dt} |z|^2 = \frac{d}{dt} z^2 = 2z \dot{z} = 2z[f(x(t), t) - f(y(t), t)] \]
at any \( t \). Then notice
\[ 2zz = 2z[f(x(t), t) - f(y(t), t)] \leq |2z \dot{z} = 2z[f(x(t), t) - f(y(t), t)] \leq 2|z|L|z| = 2L|z|^2. \]
This gives a differential inequality like the one from the previous exercise. Therefore, we conclude that for \( t > t_0 \) we have
\[ |z(t)|^2 \leq |z(t_0)|^2 e^{2L(t - t_0)} = 0 \]
since \( z(t_0) = 0 \). This shows that \( x(t) = y(t) \) when \( t > t_0 \).
Now, consider the function \( g(x, t) = -f(x, t) \). If we define \( w_1(t) = x(-t) \) and \( w_2(t) = y(-t) \) then we have
\[ \dot{w}_1(t) = -\dot{x}(-t) = -f(x(-t), -t) = g(w_1(t), t) \]
and similarly
\[ \dot{w}_2(t) = g(w_2(t), t). \]
It is also straightforward to see that
\[ |g(w_1(t), t) - g(w_2(t), t)| = |f(w_1(t), -t) - f(w_2(t), -t)| \leq L|w_1(t) - w_2(t)|. \]
Therefore, by what was proved earlier, we have \( w_1(t) = w_2(t) \) whenever \( t > -t_0 \), or equivalently \( x(t) = y(t) \) whenever \( t < t_0 \) so \( x(t) = y(t) \) for all real \( t \) as desired. \( \square \)