PART II: LINEAR EQUATIONS

13. Basic concepts

13.1. Linear equations. The standard form of a second order linear equation is

\[ L[x] \equiv \ddot{x} + p(t) \dot{x} + q(t)x = g(t). \]

The map \( x \mapsto L[x] \) is a differential operation. If \( g(t) \equiv 0 \), then the equation

\[ L[x] = 0 \]

is called homogenous. The operation \( L \) has constant coefficients if \( p(t) \) and \( q(t) \) are constant functions.

The standard form of an \( n \)-th order equation is \( L[x] = g(t) \) with

\[ L[x] = x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x. \]

13.2. Initial value problem. To solve the IVP(\( t_0, x_0, \dot{x}_0 \)) for a second order equation (not necessarily linear) is to find a solution \( x(t) \) such that

\[ x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0. \]

In the case of \( n \)-th order equations, the initial conditions have the form

\[ x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0, \ldots, \quad x^{(n-1)}(t_0) = x_0^{(n-1)}. \]

Theorem. Consider a linear equation \( L[x] = g \). Suppose that \( g \) and the coefficients of \( L \) are continuous functions in some interval \( I \). Then

(i) all solutions extend to the whole interval,

(ii) every IVP with \( t_0 \in I \) has a unique solution defined on \( I \).

13.3. Linear properties. If the coefficients of \( L \) are in \( C(I) \), then we have a linear map

\[ L : C^{(n)}(I) \rightarrow C(I). \]

Denote

\[ \ker L = \{ x : L[x] = 0 \}. \]

This is the set of all solutions of the homogeneous equation.

Lemma. \( \ker L \) is a linear space of dimension \( n \).

Proof: Fix some \( t_0 \in I \) and consider the map

\[ \mathbb{R}^n \rightarrow \ker L, \quad (x_0, \dot{x}_0, \ldots, x_0^{(n-1)}) \mapsto x(t), \]

where \( x(t) \) is the solution of the corresponding IVP. Applying the theorem we see that this map is linear, 1-to-1, and "onto". \( \square \)

Recall that \( n \) elements (or vectors) \( x_1, \ldots, x_n \) of a linear space are (linearly) independent if

\[ C_1 x_1 + \cdots + C_n x_n = 0 \quad \Rightarrow \quad C_1 = \cdots = C_n = 0. \]
If the linear space has dimension \( n \), then \( n \) independent vectors form a basis: every vector \( x \) has a unique representation as a linear combination of \( x_j \)'s. Any \( n + 1 \) vectors of an \( n \) dimensional space are dependent.

Let \( x_1, \ldots, x_n \) be solutions of an \( n \)-th order homogeneous equation \( L[x] = 0 \). Then the collection \( x_1, \ldots, x_n \) is called a fundamental system of solutions if the functions \( x_j \) are independent as elements of \( \ker L \). It follows that every solution of \( L[x] = 0 \) has a unique representation as a linear combination

\[
x(t) = C_1 x_1(t) + \cdots + C_n x_n(t).
\]

**Example.** The functions \( 1, t, \ldots, t^{n-1} \) form a fundamental system of solutions of the equation \( x^{(n)} = 0 \). The general solution (≡ the set of all solutions) is the space of polynomials of degree \( <n \).

**Lemma.** If \( x_* \) is a solution of \( L[x] = g \), then the general solution of \( L[x] = g \) is \( x \in x_* + \ker L \), (or \( x = x_* + C_1 x_1 + \cdots + C_n x_n \), "particular integral" + "complementary function").

**13.4. Reduction of order.** The following substitution is quite useful.

**Theorem.** If we know some solution \( x_1 \) of a homogeneous equation \( L[x] = 0 \) of order \( n \), then the substitution \( x = x_1 y \), where \( y = y(t) \) is a new unknown function, transforms the equation \( L[x] = g \) into a linear equation of order \( n-1 \) for \( y' \).

**Proof:** Consider the case \( n = 3 \),

\[
L[x] \equiv x''' + p_1 x'' + p_2 x' + p_3 x = g,
\]

(the coefficients are not necessarily constant). We have

\[
L[x] = x_1''' y + 3x_1'' y' + 3x_1' y'' + x_1 y''''
+ p_1 x_1'' y' + 2p_1 x_1' y''
+ p_2 x_1' y'
+ p_3 x_1 y = g.
\]

The sum of the first terms is \( yL[x_1] = 0 \); the other terms do not involve \( y \). \( \square \)

### 14. The Wronskian

**14.1. Definition.** The Wronskian of two functions \( y_1(x), y_2(x) \in C^1(I) \) is the function

\[
W(x) = \begin{vmatrix}
y_1(x) & y_2(x) \\
y_1'(x) & y_2'(x)
\end{vmatrix} \in C(I).
\]

The Wronskian of \( n \) functions \( y_j \in C^{(n-1)}(I) \) is defined similarly, e.g.

\[
W = \begin{vmatrix}
y_1 & y_2 & y_3 \\
y_1' & y_2' & y_3' \\
y_1'' & y_2'' & y_3''
\end{vmatrix} \in C(I).
\]

Let us consider a linear homogeneous equation with continuous coefficients,

\[
L[y] \equiv y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0, \quad p_j \in C(I).
\]
Theorem. Let $y_1, \ldots, y_n$ be solutions of $L[y] = 0$. FAE:

(i) \{y_1, \ldots, y_n\} is a fundamental system of solutions;

(ii) $\forall x \in I, W(x) \neq 0$;

(iii) $\exists x_0 \in I, W(x_0) \neq 0$.

Proof: As we know, the map

$$\ker L \to \mathbb{R}^n : y \mapsto (y(x_0), \ldots, y^{(n-1)}(x_0))$$

is a linear isomorphism for every $x_0 \in I$.

(i)$\Rightarrow$(ii) If the solutions $y_j$ are independent, then the vectors of initial conditions are independent in $\mathbb{R}^n$, and therefore $W(x_0) \neq 0$ for all $x_0$.

(iii)$\Rightarrow$(i) If $W(x_0) \neq 0$, then the initial vectors are independent, and so are the solutions. \qed

The most interesting part of this theorem is the equivalence of (ii) and (iii).

Examples. (a) The Wronskian of $y_1(x) = x$ and $y_2(x) = \sin x$ is zero at $x = 0$ but not identically. The Wronskian of $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ is identically zero but the functions are independent.

Conclusion: these pairs of functions can not be solutions of the same second order linear equation with continuous coefficients.

(b) The Wronskian of exponential functions $e^{\lambda_j x}$ evaluated at 0 is the so called Vandermonde determinant

$$W(0) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = \prod_{j<k} (\lambda_j - \lambda_k).$$

14.2. Abel’s formula. We don’t need to solve the differential equation to compute the Wronskian of its fundamental system of solutions.

Theorem. Let $W(x)$ be the Wronskian of some fundamental system of solutions of $L[y] = 0$. Then

$$W(x) = W(x_0) \exp \left\{ - \int_{x_0}^x p_1(s) ds \right\}.$$

Proof: We will verify the statement for second order equations

$$y'' + p(x)y' + q(x)y = 0. \quad (14.1)$$

It is sufficient to show that the Wronskian satisfies the equation

$$W' = -p(x)W.$$
We have
\[
W' = \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}
\]
\[
= 0 + \begin{vmatrix} y_1 & y_2 \\ -py'_1 & -py'_2 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ -qy_1 & -qy_2 \end{vmatrix}
\]
\[
= -p \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -pW.
\]

\[\square\]

**Corollary.** Let \( y_1 \) be a (known) solution of the 2nd order equation (14.1). Define
\[
W(x) = \exp \left\{ -\int_{x_0}^{x} p(s)ds \right\}.
\]
Then
\[
y_2 = y_1 \int \frac{W}{y_1^2}
\]
(14.2)
is also a solution, and \( y_1, y_2 \) are independent.

**Proof:** According to the above theorem, there is an independent solution \( y_2 \) such that \( W \) is the Wronskian of \( y_1, y_2 \). To find \( y_2 \) consider the formula
\[
y_1y_2' - y_1'y_2 = W
\]
as a first order linear equation for \( y_2(x) \); the coefficients \( y_1, y_1' \), and \( W \) are known functions. We have
\[
\left( \frac{y_2}{y_1} \right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}
\]
which gives us (14.2) \[\square\]

**Exercise:** Strictly speaking, the formula (14.2) makes sense only on the intervals where \( y_1 \neq 0 \). Interpret the formula in the case where \( y_1 \) has zeros. E.g. \( y'' + y = 0 \), \( y_1(t) = \sin t \).

**Example.** Consider the equation \( x^2(1 + x)y'' - 2y = 0 \). One can verify that \( y_1(x) = 1 + x^{-1} \) is a solution. Find the general solution and determine the domain of the maximal solution of the IVP\((-2; 0, 1)\).

According to the general theory of linear equations, the IVP solution extends at least to \((-\infty, -1)\) (but in principle it may exist on a larger interval.) Since the Wronskian is constant, we find
\[
y_2 = y_1 \int \frac{1}{y_1^2} = y_1(x) \int \frac{x^2dx}{(1+x)^2} = 1 + x - \frac{1}{x} - \left(1 + \frac{1}{x}\right) \log(1 + x)^2.
\]
The general solution is \( y = C_1y_1 + C_2y_2 \), and we have \( C_2 \neq 0 \) for the IVP solution. The IVP solution is not differentiable at \( x = -1 \), so its domain is \((-\infty, -1)\).
15. Homogeneous equations with constant coefficients

\[ L[x] \equiv a_0 x^{(n)} + \cdots + a_n x = 0, \quad a_j \in \mathbb{R}. \]

15.1. **Characteristic polynomial.** Denote

\[ P(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n, \]

so (symbolically)

\[ L = P(D), \quad D = \frac{d}{dt}. \]

The polynomial \( P \) has exactly \( n \) roots ("eigenvalues")

\[ \lambda_1, \ldots, \lambda_n, \quad P(\lambda_j) = 0, \]

which may be complex and/or multiple, and we have

\[ P(\lambda) = a_0(\lambda - \lambda_1) \cdots (\lambda - \lambda_n). \]

Since the characteristic polynomial \( P \) has real coefficients, complex roots appear in *conjugate pairs*, i.e. if \( \lambda = \alpha + i\omega \) is an eigenvalue, then \( \lambda = \alpha - i\omega \) is also an eigenvalue.

**Lemma.** \( L[e^{\lambda t}] = P(\lambda)e^{\lambda t} \).

**Proof:** If \( x(t) = e^{\lambda t} \), then \( x^{(k)}(t) = \lambda^k e^{\lambda t} \).

**Corollary.** A real number \( \lambda \) is a root of the characteristic polynomial if and only if \( x(t) = e^{\lambda t} \) is a solution of the homogeneous equation.

15.2. **The case of distinct real roots.**

**Theorem.** If the roots of the characteristic polynomial are all real and distinct, then the general solution of the homogeneous equation is

\[ x(t) = C_1 e^{\lambda_1 t} + \cdots + C_n e^{\lambda_n t}. \]

**Proof:** The exponential functions with distinct exponents are linearly independent. Indeed, suppose

\[ C_1 e^{\lambda_1 t} + \cdots + C_n e^{\lambda_n t} \equiv 0 \]

and \( \lambda_1 < \cdots < \lambda_n \). If \( C_n \neq 0 \), then the LHS divided by \( C_n e^{\lambda_n t} \) converges to 1 as \( t \to \infty \), a contradiction. If \( C_n = 0 \), we repeat this argument with \( C_{n-1} \) instead of \( C_n \), etc.
15.3. **Second order equations.** There are 3 possible cases:

(i) the roots $\lambda_1, \lambda_2$ are real and distinct;

(ii) $\lambda_1 = \lambda_2 \in \mathbb{R}$;

(iii) $\lambda_{1,2} = \alpha \pm i\omega$ with $\omega \neq 0$.

**Theorem.** The general solution in the above three cases is

(i) $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$;

(ii) $x(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$;

(iii) $x(t) = C_1 e^{\alpha t} \cos \omega t + C_2 e^{\alpha t} \sin \omega t$.

**Proof:**

(i) is clear.

(ii) is the limiting case of (i) as $\lambda_2 \to \lambda_1$:

$$
\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \to \frac{d}{d\lambda} \bigg|_{\lambda = \lambda_1} e^{\lambda t} = t e^{\lambda_1 t}.
$$

Alternatively, we can apply the reduction of order method. The equation is $x'' - 2\lambda_1 x' + \lambda_1^2 x = 0$, and $x_1 = e^{\lambda_1 t}$ is a solution. Substitution $x = x_1 y$ gives the equation $y'' = 0$, so we can take $y(t) = t$ and get the second solution $x_2 = t e^{\lambda_1 t}$.

(iii) easy to verify; also see the next section.

15.4. **Difference equations.** *Fibonacci numbers*

$$
0 \ 1 \ 1 \ 2 \ 3 \ 5 \ 8 \ldots
$$

satisfy the second order linear difference (or recursive) equation

$$
x_n = x_{n-1} + x_{n-2}.
$$

(Fibonacci’s interpretation: $x_n$ is the population of rabbits in the $n$-th generation.)

The linear equation is homogeneous. Let us look for solutions of the form

$$
x_n = a^n,
$$

(discrete exponentials). Then $a$ has to satisfy the equation

$$
a^n = a^{n-1} + a^{n-2}, \quad a^2 - a - 1 = 0.
$$

We have two roots

$$
a_{1,2} = \frac{1 \pm \sqrt{5}}{2},
$$

($a_1$ is the "golden ratio", see Problem 22.3). The general solution to the difference equation is

$$
x_n = C_1 a_1^n + C_2 a_2^n.
$$

From the initial conditions

$$
x_0 = 0, \quad x_1 = 1,
$$

we find the values

$$
C_1 = -C_2 = \frac{1}{\sqrt{5}}.
$$

This method works for all linear homogeneous recursive equations with constant coefficients. See Examples 22.1 and 22.2 for the cases of repeated and complex roots.
16. Complex eigenvalues

16.1. Complex exponentials. A complex-valued function of one real variable,

\[ z(t) = x(t) + iy(t), \quad t \in \mathbb{R}, \]

(the functions \( x(t) \) and \( y(t) \) are real-valued) is differentiable if both \( x(t) \) and \( y(t) \) are differentiable, and in this case

\[ z' = x' + iy'. \]

**Example.** Let \( \lambda = \alpha + i\omega \). By definition,

\[ e^{\lambda t} = e^{\alpha t} \cos(\omega t) + ie^{\alpha t} \sin(\omega t), \quad (16.1) \]

and it is easy to check that

\[ \frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}. \]

A more conceptual approach to the formula (16.1) is to introduce the complex exponential function

\[ e^{\lambda} = \sum_{n \geq 0} \frac{\lambda^n}{n!}, \quad \lambda \in \mathbb{C}, \quad (16.2) \]

If \( \lambda = \alpha \) is real, then we get the usual (real) exponential \( e^{\alpha} \), and if \( \lambda = i\omega \) is imaginary, then

\[ e^{i\omega} = \cos \omega + i \sin \omega \]

because the real and imaginary parts of the series (16.2) are Taylor’s series of cosine and sine functions respectively. The series (16.2) converges in the whole complex plane, and one can verify (by multiplication of the series) the functional equation

\[ e^{\lambda_1 + \lambda_2} = e^{\lambda_1} e^{\lambda_2}. \]

In particular, for \( \lambda = \alpha + i\omega \) we have

\[ e^{\lambda t} = e^{\alpha t} e^{i\omega t} = e^{\alpha t} \cos \omega t + ie^{\alpha t} \sin \omega t, \]

which is the same as (16.1).

16.2. Complex solutions. Consider the equation

\[ L[z] \equiv P(D)[z] = 0 \]

for complex-valued functions \( z = z(t) \). Similarly to the real case, we have

\[ \forall \lambda \in \mathbb{C}, \quad L[e^{\lambda t}] = P(\lambda) e^{\lambda t}, \]

which proves

**Lemma.** \( z(t) = e^{\lambda t} \) is a solution iff \( P(\lambda) = 0 \).

We also have the following statement concerning linear equations \( L[z] = 0 \) with real coefficients.

**Lemma.** \( z(t) = z(t) + iy(t) \) is a (complex) solution iff both \( x(t) \) and \( y(t) \) are (real) solutions.
Proof: Since the coefficients of the equation are real, we have $L[z] = L[x] + iL[y]$, so $L[z] = 0$ iff $L[x] = 0$ and $L[y] = 0$. \hfill \Box$

Combining the two lemmas, we derive

**Theorem.** Consider the equation $L[x] = 0$ with constant real coefficients, and suppose that all roots of the characteristic polynomial are distinct. Let $\lambda_1, \ldots$ be the real roots, and let $\alpha_1 \pm i\beta_1, \ldots$ be the complex roots. Then the collection of functions

$$e^{\lambda_1 t}, \ldots; \quad e^{\alpha_1 t} \cos(\beta_1 t), \quad e^{\alpha_1 t} \sin(\beta_1 t), \ldots$$

is a fundamental system of (real-valued) solutions.

**Note.** In the case of distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ (real or complex), the general complex-valued solution is

$$z(t) = \sum_{j=1}^{n} C_j e^{\lambda_j t},$$

where $C_j$'s are arbitrary complex numbers.

16.3. **General homogeneous equations with constant coefficients.** Consider the equation $L[x] = 0$ with real constant coefficients. If $\lambda_j$ is a repeated real root of multiplicity $k$,

$$P(\lambda) = (\lambda - \lambda_j)^k \tilde{P}(\lambda), \quad \tilde{P}(\lambda_j) \neq 0,$$

then we add the following functions to the list (16.3) of the last theorem:

$$te^{\lambda_j t}, \ldots, \quad t^{k-1}e^{\lambda_j t}.$$ 

If $\alpha_j \pm i\beta_j$ is the conjugate pair of complex roots of multiplicity $k$, then we add the functions

$$te^{\alpha_j t} \cos(\beta_j t), \quad te^{\alpha_j t} \sin(\beta_j t), \ldots, \quad t^{k-1}e^{\alpha_j t} \cos(\beta_j t), \quad t^{k-1}e^{\alpha_j t} \sin(\beta_j t).$$

Using reduction of order, it is not difficult to justify the following statement.

**Theorem.** The resulting collection of functions is a fundamental system of solutions of the equation $L[x] = 0$.

**Example.** Let $P(\lambda) = \lambda^3(\lambda^2 + 1)^2$. The roots are

$$0, \ 0, \ i, \ -i, \ i, \ -i.$$ 

We have the following fundamental system of solutions:

$$1, \ t, \ t^2, \ \cos t, \ \sin t, \ t \cos t, \ t \sin t.$$ 

The general solution of the equation $x^{(7)} + 2x^{(5)} + x^{(3)} = 0$ is therefore

$$x(t) = C_1 + C_2 t + C_3 t^2 + C_4 \cos t + C_5 \sin t + C_6 t \cos t + C_7 t \sin t.$$
17. Inhomogeneous linear equations

We will discuss here how to solve equations

\[ L[x] = g. \]

We have two general methods: reduction of order (Section 15.4) and variation of constants. These methods work for all linear equations, not necessarily with constant coefficients. The third method, the method of undetermined coefficients, is often the most efficient but it applies only to equations with constant coefficients and to functions \( g \) of special type (quasi polynomials). Also, see Section *** below for the discussion of the important Laplace method.

17.1. Variation of constants. We will show that if we know the general solution of the homogeneous equation \( L[x] = 0 \), then we can solve any inhomogeneous equation \( L[x] = g \). For simplicity, we will only discuss the case of 2d order equations:

\[ L[x] = x'' + p(t)x' + q(t)x. \]

Let \( x_1 \) and \( x_2 \) be two independent solutions of \( L[x] = 0 \). We look for a particular solution of the inhomogeneous equation of the form

\[ x_* = C'_1(t)x_1 + C'_2(t)x_2. \]

We have

\[ x'_* = (C'_1x'_1 + C'_2x'_2) + (C''_1x_1 + C''_2x_2), \]

and if the functions \( C'_1 \) and \( C'_2 \) are such that

\[ C'_1x_1 + C'_2x_2 \equiv 0, \]

then

\[ x''_* = (C''_1x'_1 + C''_2x'_2) + (C'_1x'_1 + C'_2x'_2), \]

so

\[ L[x_*] = C'_1x'_1 + C'_2x'_2. \]

Thus \( x_* \) is a particular solution if \( C'_1 \) and \( C'_2 \) satisfy the system of linear algebraic equations

\[ C'_1x_1 + C'_2x_2 = 0, \quad C'_1x'_1 + C'_2x'_2 = g(t). \]

Solving this system:

\[ C'_1 = \begin{vmatrix} \frac{x_2}{g} & x_2 \\ \frac{g}{x_2} & x'_2 \end{vmatrix} = \frac{gx_2}{W}, \quad C'_2 = \begin{vmatrix} x_1 & 0 \\ x'_1 & g \end{vmatrix} = \frac{gx_1}{W}, \quad W := \begin{vmatrix} x_1 & x'_1 \\ x_2 & x'_2 \end{vmatrix}, \]

we derive

\[ x_* = -x_1 \int \frac{gx_2}{W} \, dt + x_2 \int \frac{gx_1}{W} \, dt. \]
17.2. Undetermined coefficients. We will now assume that \( L[\cdot] \) has constant coefficients and \( g \) is a quasi polynomial.

By definition, a quasi polynomial is a linear combination of the functions of the following form:

\[
    t^m e^{\mu t}, \quad t^m e^{\alpha t} \cos(\omega t), \quad t^m e^{\alpha t} \sin(\omega t).
\]

(17.1)

Here \( m \) is a non-negative integer, and \( \mu, \alpha, \omega \) are real. We will write \( \mu = \alpha + i\omega \) in the last two cases in (17.1).

We will show that the equation \( L[x] = g \) has a quasi polynomial solution. Because of the following (obvious) lemma, we can consider the functions (17.1) separately.

**Lemma.** If \( g = g_1 + g_2 \), and if \( x_1 \) and \( x_2 \) are solutions of \( L[x] = g_1 \) and \( L[x] = g_2 \) respectively, then \( x = x_1 + x_2 \) is a solution of \( L[x] = g \).

Let \( g \) be one of the functions (17.1), and let \( k \) denote the multiplicity of \( \mu \) as a root of the characteristic polynomial; e.g. \( k = 0 \) if \( \mu \) is not an eigenvalue.

**Theorem.** (i) If \( \mu \) is real (i.e. \( g = t^m e^{\mu t} \)), then there is a solution of the form

\[
    x_*(t) = t^k \text{(some polynomial of degree } m) \ e^{\mu t}.
\]

(ii) If \( \mu \) is complex, then there is a solution of the form

\[
    x_*(t) = t^k \text{(some polynomial of degree } m) \ e^{\alpha t} \cos(\omega t) \\
    + t^k \text{(some other polynomial of degree } m) \ e^{\alpha t} \sin(\omega t).
\]

**Proof:** We’ll only discuss the simpler case \( k = 0 \), i.e. \( P(\mu) \neq 0 \). If \( \mu \) is real, then we denote

\[
    E_{\mu,m} = \{ \text{real polynomials of degree } \leq m \} \cdot e^{\mu t}.
\]

This is a linear space (over \( \mathbb{R} \)) of dimension \( m + 1 \). It is enough to show that the linear map \( L: x \mapsto L[x] \) acts and is invertible in \( E_{\mu,m} \). Let us compute the matrix of

\[
    D \equiv \frac{d}{dt}: E_{\mu,m} \to E_{\mu,m}
\]

in the basis

\[
    e_j = t^j e^{\mu t}, \quad (0 \leq j \leq m).
\]

Since

\[
    D e_j = j t^{j-1} e^{\mu t} + \mu t^j e^{\mu t} = j e_{j-1} + \mu e_j,
\]

we have

\[
    D = \begin{pmatrix}
    \mu & 1 & 0 & \cdots \\
    0 & \mu & 2 & \cdots \\
    0 & 0 & \mu & \cdots \\
    \vdots & \vdots & \vdots & \ddots
    \end{pmatrix}.
\]

It follows that

\[
    L \equiv P(D) = \begin{pmatrix}
    P(\mu) & * & * & \cdots \\
    0 & P(\mu) & * & \cdots \\
    & & & \ddots
    \end{pmatrix}.
\]
Since $P(\mu) \neq 0$, the eigenvalues of the last matrix are non-zero, and the operator $L : E_{\mu,m} \rightarrow E_{\mu,m}$ is invertible. If $\mu$ is complex, we consider complex polynomials in the definition of $E_{\mu,m}$ and use the same argument.

17.3. Examples.

(i) $x'' + x = t^2$.
We have $\mu = 0$, which is not a root of the characteristic polynomial $\lambda^2 + 1$, so $k = 0$. There is a solution of the form

$$x(t) = At^2 + Bt + C.$$ 

Let’s find the (undetermined) coefficients. We need to satisfy

$$x'' + x = At^2 + Bt + (2A + C) = t^2,$$

so

$$A = 1, \quad B = 0, \quad C = -2A = -2.$$ 

Thus we get a particular solution $x(t) = t^2 - 2$. The general solution is then

$$x(t) = C_1 \cos t + C_2 \sin t + t^2 - 2.$$ 

(ii) $x'' + x = e^{2t}$.
We have $\mu = 2$, which is not a root of the characteristic polynomial, so $k = 0$. There is a solution of the form $x(t) = Ae^{2t}$. We find $A = 1/5$.

(iii) $x'' + x = te^{-t}$.
There is a solution of the form $x(t) = (At + B)e^{-t}$.

(iv) $x'' + x = t^3 \sin t$.
Now $\alpha = 0$, $\omega = 1$, and $\mu = i$, which is a root of multiplicity $k = 1$. There is a solution of the form

$$x(t) = (At^4 + Bt^3 + Ct^2 + Dt) \cdot \sin t + (\check{A}t^4 + \check{B}t^3 + \check{C}t^2 + \check{D}t) \cdot \cos t.$$ 

(v) $x^{(4)} + 4x'' = \sin 2t + te^t + 4$.
The characteristic polynomial is

$$P(\lambda) = \lambda^4 + 4\lambda^2 = \lambda^2(\lambda^2 + 4), \quad \lambda_{1,2} = 0, \quad \lambda_3 = 2i, \quad \lambda_4 = -2i.$$ 

We have

$$g_1 = \sin 2t \quad \Rightarrow \quad x_1 = t(A\sin 2t + B\cos 2t);$$
$$g_2 = te^t \quad \Rightarrow \quad x_2 = e^t(Ct + D);$$
$$g_1 = 4 \quad \Rightarrow \quad x_3 = Et^2,$$

and so there is a solution of the form

$$x(t) = At \sin 2t + Bt \cos 2t + Cte^t + De^t + Et^2.$$
17.4. **Inhomogeneous linear difference equations.** Example: the sequence

1, 1, 6, 12, 29, 59, ...

is described by the IVP

\[ x_n = x_{n-1} + 2x_{n-2} + n, \quad x_1 = x_2 = 1. \]

The associated homogeneous equation is \( x_n = x_{n-1} + 2x_{n-2} \), and if \( x_n = a^n \) is its solution, then

\[ a^2 = a + 2; \quad a_1 = -1, \ a_2 = 2. \]

Since \( n = n1^n \) and 1 is not a root of \( a^2 = a + 2 \), we will look for a particular solution of the inhomogeneous equation of the form

\[ x_n = An + B. \]

We have

\[ An + B = An - A + B + 2An - 4A + 2B + n, \]

and therefore

\[ A = A + 2A + 1, \quad B = -A + B - 4A + 2B. \]

Thus \( A = -1/2, \ B = -5/4 \). Conclusion: the general solution is

\[ x_n = -(n + 5)/2 + (-1)^nC_1 + 2^nC_2, \]

and it remains to determine \( C_1 \) and \( C_2 \) from the initial conditions.

18. **Oscillations**

In this section we discuss applications to mechanical and electrical vibrations. We will consider 2nd order equations

\[ \ddot{x} + 2a \dot{x} + \omega_0^2 x = g(t) \quad (18.1) \]

with constant coefficients and periodic functions \( g(t) \). It is remarkable that such simple equations have many meaningful applications.

18.1. **Interpretations.**

(a) **Harmonic oscillator:** the motion of a mass \( m \) on a Hooke’s law spring with spring constant \( k > 0 \). Newton’s equation for the elongation of the spring \( x = x(t) \) is

\[ m\ddot{x} + \gamma \dot{x} + kx = g(t), \]

where \( \gamma \) is the damping coefficient, and \( g(t) \) is the external force.

(b) **Electrical circuits.** An RLC circuit consists of a **resistor** with resistance \( R \) (ohms), an **inductor** with inductance \( L \) (henrys), and a **capacitor** with capacitance \( C \) (farads).

The resistor converts electrical energy into heat or light, e.g. a light bulb element. The inductor has a special geometry such as coils which creates a magnetic field that induces a voltage drop. A typical capacitor consists of two plates separated by an insulator; charges of opposite signs will build up on the two plates.
The impressed voltage $E(t)$ is a given function of time (seconds). We consider the following characteristics of the circuit: the charge $Q(t)$ (coulombs) on the capacitor, and the current $I(t) = \dot{Q}(t)$ (amperes). By Kirchhoff,

$$E(t) = LI + RI + C^{-1}Q.$$  

($E(t)$ is equal to the sum of voltage drops across $R$, $L$, and $C$; the respective drops are $RI$ by Ohm’s law, $C^{-1}Q$ by Coulomb’s law, and $LI$ by Faraday’s law.) Thus we get equations similar to the mass-spring model:

$$E(t) = L\ddot{Q} + R\dot{Q} + C^{-1}Q,$$

or

$$\dot{E}(t) = LI + RI + C^{-1}I,$$

where $R$ is responsible for damping.

18.2. **Free vibrations:** $g = 0$ in (18.1).

(a) **Undamped case:** $\alpha = 0$. The general solution is

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t = M \cos (\omega_0 t - \phi).$$

We have a periodic motion with period equal to $2\pi/\omega_0$; $\omega_0$ is the natural frequency, $M > 0$ is the amplitude of the motion (maximal displacement), and $\phi$ is the phase angle.

(b) **Damped case.** The eigenvalues are

$$\lambda_{1,2} = -a \pm \sqrt{a^2 - \omega_0^2}.$$  

There are no oscillations in the over-damped case $a^2 \geq \omega_0^2$. Otherwise ("under-damping"), we have complex eigenvalues

$$\lambda_{1,2} = -a \pm i\mu,$$

and the motion is a damped vibration:

$$x(t) = Re^{-at} \cos(\mu t - \delta);$$

$\mu$ is called quasi frequency and $2\pi/\mu$ quasi period.

18.3. **Forced vibrations:** undamped case. Suppose the exterior force $g$ in (18.1) is $T$-periodic,

$$g(t + T) = g(t),$$

and denote $\omega = 2\pi/T > 0$, the forcing frequency. We will study the question of existence and uniqueness of periodic solutions and also the question of their stability. We only discuss the case

$$g(t) = A \cos(\omega t - \delta),$$

and refer to Arnol’d for general theory. Assume first that there is no damping, so the equation is

$$\ddot{x} + \omega_0^2 x = A \cos(\omega t - \delta).$$
**Theorem.** (i) If the ratio $\omega_0/\omega$ is not a rational number, then there exists a unique periodic solution. The period of the periodic solution is $T$.

(ii) If $\omega_0/\omega$ is a rational number but $\omega \neq \omega_0$, then all solutions are periodic.

(iii) If $\omega = \omega_0$, then every solution is unbounded and non-periodic.

The case (iii) is called (pure) resonance: "bounded input, unbounded output."

**Proof:** For simplicity assume $\delta = 0$. If $\omega \neq \omega_0$, then the general solution is

$$x(t) = C_1 \cos \omega_0 t + C_1 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t,$$

(again, use undetermined coefficients). The choice $C_1 = C_2 = 0$ gives a periodic solution. All other solutions are not periodic unless the frequencies are commensurable. In the resonance case, we have

$$x(t) = C_1 \cos \omega_0 t + C_1 \sin \omega_0 t + \frac{A}{2\omega_0} t \sin \omega_0 t,$$

(again, use undetermined coefficients). \(\square\)

**Comments.**

(a) The results in the case of a general periodic exterior force are similar. In particular, if the frequency of the exterior force is equal to the natural frequency, then all solutions are unbounded. Soldiers used to break step when crossing a bridge to eliminate the periodic force of their marching that could resonate with a natural frequency of the bridge.

(b) **Beats.** In the non-resonance case, every solution is the sum of two periodic functions,

$$x(t) = A \cos (\omega_0 t + \alpha) - B \cos (\omega t + \beta).$$

The graphs of such functions could be quite intriguing. For example, consider the case where the amplitudes are equal, $A = B$, (or almost equal), and the frequencies are not very different in the sense that

$$|\omega - \omega_0| \ll |\omega + \omega_0|.$$  

(Think of two flutes playing slightly out of tune with each other.) We have (for $\alpha = \beta = 0$, otherwise up to a phase shift)

$$x(t) = 2A \sin \frac{\omega - \omega_0}{2} t \sin \frac{\omega + \omega_0}{2} t.$$  

The first term oscillates slowly compared to the second one. We can think of the function

$$\left|2A \sin \frac{\omega - \omega_0}{2} t\right|$$

as a slowly changing amplitude of the vibrations with frequency $|\omega + \omega_0|/2$. We say that the solution exhibits the phenomenon of beats.

(c) Example of case (ii):

$$\ddot{x} + 36x = 3 \sin 4t.$$  

Every solution is the sum of a $2\pi/6$-periodic function and a $2\pi/4$-periodic function, so every solution is $\pi$-periodic.
Relation to Lissajous’ curves: consider the orbit of the motion \( \{x(t), \dot{x}(t)\} \) in the "phase space" \( \mathbb{R}^2 = \{(x, \dot{x})\} \). For example, draw the curve

\[
x(t) = -\frac{1}{10} \sin 6t + \frac{3}{20} \sin 4t, \quad \dot{x}(t) = -\frac{3}{5} \cos 6t + \frac{3}{5} \cos 4t,
\]

which is the orbit of the IVP(0;0,0).

18.4. Forced vibrations: damped case.

\[\ddot{x} + 2a \dot{x} + \omega_0^2 x = A \cos(\omega t - \delta).\]

**Theorem.** If \( a > 0 \), then there is a unique periodic solution \( x_*(t) \). This solution has period \( 2\pi/\omega \) and is a steady state.

The last sentence means that \( x_*(t) \) is globally asymptotically stable: for any solution \( x(t) \), we have

\[|x(t) - x_*(t)| \to 0 \quad \text{as} \quad t \to \infty\]

(convergence is exponentially fast). In other words, no matter what initial conditions are, we asymptotically get the same function \( x_*(t) \). Note that there are no steady states in the undamped case — the effect of the initial conditions would persist at all times.

**Proof:** The eigenvalues are \( \lambda_{1,2} = -a \pm \sqrt{a^2 - \omega_0^2} \). Clearly, \( \Re \lambda_1 < 0 \) and \( \Re \lambda_2 < 0 \), in particular \( i\omega \) is not an eigenvalue. By the method of undetermined coefficients, there is a particular solution \( x_*(t) \) of the form \( B \cos(\omega t - \phi) \). On the other hand, every "complementary function" (solution to the associated homogeneous equation) is exponentially small as \( t \to +\infty \), so the general solution is

\[x(t) = B \cos(\omega t - \phi) + o(1), \quad t \to +\infty.\]

This theorem generalizes to arbitrary linear systems with constant coefficients such that all eigenvalues have negative real parts. If the non-homogeneous term is periodic, then there is a unique periodic solution, which is a steady state.

**Terminology:** the periodic force \( g(t) \) is the input, and the steady state is the output or forced response of the system. The \( o(1) \) part of the solution is the transient solution. The ratio of the amplitudes \( G = B/A \) is the gain.

**Exercise:** show

\[G = \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4a^2\omega^2}}.\]  

\[\text{(18.2)}\]

**Proof:** If \( x = \tilde{B}e^{i\omega t} \) with \( |\tilde{B}| = B \), then \( L[x] = \tilde{A}e^{i\omega t} \) with \( |\tilde{A}| = A \) and \( p(i\omega)\tilde{B} = \tilde{A} \).

We have

\[G = \frac{|\tilde{B}|}{|\tilde{A}|} = \frac{1}{|p(i\omega)|} = \frac{1}{|\omega_0^2 - \omega^2| + 2a\omega i}|.\]

The gain can be large if the frequencies are almost equal and the damping is small, the case of ”practical resonance”.

\[\square\]
Applications. (a) For a given circuit or spring \((\omega_0, a)\) are fixed), the gain is a function of \(\omega\), \(G = G(\omega)\). Sometimes we want to find the "resonant frequency" \(\omega_r\) for which the gain is maximal. Assuming \(a < \omega_0^2\) (small damping), we have

\[
\omega_r^2 = \omega_0^2 - 2a^2, \quad G_{\text{max}} = \frac{1}{2a\sqrt{\omega_0^2 - a^2}},
\]

(differentiate the expression under the square root in (18.2) with respect to \(\omega^2\)).

(b) Tuning. Consider the equation.

\[
L\ddot{x} + R\dot{x} + C^{-1}x = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t,
\]

where \(L\) and \(R\) are fixed but \(C\) is a tuning parameter. Think of two radio station broadcasting on different frequencies. We want to tune the radio to one of the stations by reducing the amplitude of the other frequency. The output (up to phase shifts) is

\[
x_\ast(t) = G_1 A_1 \cos \omega_1 t + G_2 A_2 \cos \omega_2 t,
\]

where the gains \(G_1\) and \(G_2\) are given by (18.2). By tuning \(C^{-1} \approx L\omega_1\) we can often make \(G_2 \ll G_1\).

19. Linear systems with constant coefficients

\[
\dot{x} = Ax + g(t), \quad x(t) \in \mathbb{R}^n.
\]

Here \(A\) is a constant \(n \times n\) matrix, \(g(t)\) is a given vector-valued function, and \(x(t)\) is the unknown vector-valued function,

\[
x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}.
\]

19.1. Converting the system into a single \(n\)-th order linear equation. This is usually the most practical method of finding an explicit solution, especially in the inhomogeneous case. Let us consider the 2D system,

\[
\begin{cases}
\dot{x} = ax + by + f(t) \\
\dot{y} = cx + dy + h(t)
\end{cases},
\]

i.e.

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g = \begin{pmatrix} f \\ h \end{pmatrix}.
\]

To solve the system, do the following:

(1) use the first equation to express \(y\) in terms of \(\dot{x}, x, t\), namely

\[
y = \frac{\dot{x}}{b} - \frac{ax}{b} - \frac{f(t)}{b}; \quad (19.1)
\]

(2) find \(\dot{y}\) in terms of \(\ddot{x}, \dot{x}, t\) by differentiating (19.1);

(3) use the second equation of the system and the expressions for \(y\) and \(\dot{y}\) to derive a 2nd order equation for \(x(t)\);

(4) solve this 2nd order equation and find \(x(t)\);
(5) use (19.1) to find \( y(t) \).

[See Examples 26.1, 26.2 in the text.]

19.2. **Vector exponentials.** This method applies to homogeneous systems

\[ \dot{x} = Ax, \quad x(t) \in \mathbb{R}^n. \]

We look for solutions of the form

\[ x(t) = e^{\lambda t}w, \quad (19.2) \]

where \( \lambda \) is a real number and \( w \) is a constant vector in \( \mathbb{R}^n \). Since

\[ \frac{d}{dt}(e^{\lambda t}w) = \lambda e^{\lambda t}w, \quad A(e^{\lambda t}w) = e^{\lambda t}Aw, \]

we see that (19.2) is a solution if and only if

\[ \lambda w = Aw, \]

i.e. if and only if \( \lambda \) is an eigenvalue of \( A \) and \( w \) is a corresponding eigenvector.

Recall that the eigenvalues of \( A \) are the roots of the characteristic polynomial

\[ P(\lambda) = \det(\lambda I - A), \]

where \( I \) is the identity matrix. We have \( n \) eigenvalues, real or complex, possibly multiple. For simple eigenvalues, the eigenvectors (in \( \mathbb{C}^n \)) are uniquely determined up to a scalar multiplicative constant. Since the matrix \( A \) is real, we can choose \( w \in \mathbb{R}^n \) if the eigenvalue is real. Eigenvectors corresponding to distinct eigenvalues are linearly independent.

**Theorem.** Suppose the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( A \) are real and distinct, and let \( w_1, \ldots, w_n \in \mathbb{R}^n \) be corresponding eigenvectors. Then

\[ x(t) = C_1 e^{\lambda_1 t}w_1 + \cdots + C_n e^{\lambda_n t}w_n, \quad C_j \in \mathbb{R}, \]

is the general solution of the system \( \dot{x} = Ax \).

**Proof:** The set of solutions is a linear space of dimension \( n \) (same argument as in the case of higher order linear differential equations, see Section 15 of the notes). The \( n \) solutions \( e^{\lambda_j t}w_j \) are linearly independent, and therefore form a basis in the space of solutions. \( \square \)

**Complex eigenvalues.** Let

\[ w = u + iv, \quad u \equiv \Re w \in \mathbb{R}^n, \quad u \equiv \Im w \in \mathbb{R}^n, \]

be an eigenvector corresponding to \( \lambda = \alpha + i\omega \). Then

\[ z(t) = e^{\lambda t}w = e^{\alpha t}(\cos \omega t + i \sin \omega t)(u + iv) \]

is a solution of the complexified equation

\[ \dot{z} = Az, \quad z(t) \in \mathbb{C}^n. \]

Since \( A \) is real, we get two real solutions:

\[ \Re z(t) = e^{\alpha t}[\cos \omega t \ u - \sin \omega t \ v], \]
\[ \Im z(t) = e^{\alpha t}[\sin \omega t \ u + \cos \omega t \ v]. \]
If \( n = 2 \) and if the eigenvalues are complex, then the general solution of \( \dot{x} = Ax \) is
\[ x(t) = C_1 \Re z(t) + C_2 \Im z(t). \]

Note. The characteristic polynomial has real coefficients, so complex eigenvalues appear in conjugate pairs and \( \bar{w} := u - iv \) is an eigenvector corresponding to \( \bar{\lambda} = \alpha - i\omega \). The complex solution \( e^{\bar{\lambda}t} \bar{w} \) produces the same real solutions as \( e^{\lambda t} w \).

20. THE EXPONENTIAL OF A MATRIX

20.1. Definition. Let \( A \) be a constant \( n \times n \) matrix. Consider the following IVP for the unknown matrix-valued function \( X(t) \):
\[ \dot{X} = AX, \quad X(0) = I \quad \text{(identity matrix).} \quad (20.1) \]
If \( x_{jk}(t) \) are the matrix elements of \( X(t) \), then
\[ \dot{X}(t) = \begin{pmatrix} \dot{x}_{11}(t) & \ldots & \dot{x}_{1n}(t) \\ \ldots & \ldots & \ldots \\ \dot{x}_{n1}(t) & \ldots & \dot{x}_{nn}(t) \end{pmatrix}, \]
so (20.1) is a linear system in \( \mathbb{R}^{n^2} \). By definition,
\[ e^{tA} = X(t), \quad \text{in particular} \quad e^A = X(1). \]

Theorem.
\[ e^{tA} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k = \lim_{k \to \infty} \left( I + \frac{tA}{k} \right)^k. \]

Remarks.
(a) Relate to Picard’s and Euler’s approximations respectively, see Section 6 of the notes.
(b) In the above statement, the limit is understood in terms of the limits of all matrix elements; same for the infinite sum.
(c) It is not difficult to justify the following formal computations:
\[ \frac{d}{dt} \left( \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k \right) = \sum_{k=0}^{\infty} \frac{d}{dt} \frac{A^k}{k!} t^k = A + \frac{2t}{2!} A^2 + \frac{3t^2}{3!} A^3 + \cdots = A \left( \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k \right), \]
and
\[ \frac{d}{dt} \left[ \lim \left( I + \frac{tA}{k} \right)^k \right] = \lim \frac{d}{dt} \left[ \left( I + \frac{tA}{k} \right)^k \right] = A \left[ \lim \left( I + \frac{tA}{k} \right)^{k-1} \right] \frac{A}{k} = A \left[ \lim \left( I + \frac{tA}{k} \right)^k \right]. \]
(d) It is not difficult to prove that if \( AB = BA \) (which is not always the case!), then
\[ e^{A+B} = e^A e^B, \]
in particular,
\[ (e^A)^{-1} = e^{-A}. \]
20.2. Solving homogeneous and inhomogeneous systems of equations.

**Theorem.** For every \( x_0 \in \mathbb{R}^n \), the vector-valued function 
\[ x(t) = e^{At}x_0 \]
is the solution of the IVP 
\[ \dot{x} = Ax, \quad x(0) = x_0. \]

**Proof:**
\[ \dot{x}(t) = \frac{d}{dt} [e^{At}x_0] = \left[ \frac{d}{dt}e^{At} \right] x_0 = Ae^{At}x_0 = Ax(t). \]

\[ \square \]

**Variation of constants.** Consider now the inhomogeneous system 
\[ \dot{x} = Ax + g(t), \quad x(t) \in \mathbb{R}^n. \]

Since \( e^{tA}C \) with \( C \in \mathbb{R}^n \) is the general solution of the associated homogeneous system of equations, we look for a particular solution [of the inhomogeneous system] of the form 
\[ x(t) = e^{tAC}(t), \]
where \( C(t) \) is the unknown vector-valued function. We have 
\[ \dot{x} = Ae^{tAC} + e^{tAC}\dot{C}, \]
and this has to be equal to 
\[ Ax + g = Ae^{tAC} + g, \]
so \( C(t) \) satisfies 
\[ e^{tAC}\dot{C} = g, \quad i.e. \quad \dot{C} = e^{-tA}g. \]

20.3. Computation of \( e^{tA} \).

(a) **Diagonal matrices:**
\[ A = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \implies e^{tA} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix}. \]

Proof:
\[ A^k = \begin{pmatrix} \lambda_1^k & 0 & \ldots & 0 \\ 0 & \lambda_2^k & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n^k \end{pmatrix}. \]

(b) **Nilpotent matrices:**
\[ N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies e^{tN} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]
Proof:

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\quad \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\quad N^4 = N^5 = \ldots = 0.
\]

(c) Jordan cells:

\[
A = \begin{pmatrix}
\lambda_0 & 1 & 0 & 0 \\
0 & \lambda_0 & 1 & 0 \\
0 & 0 & \lambda_0 & 1 \\
0 & 0 & 0 & \lambda_0
\end{pmatrix} \implies e^{tA} = e^{\lambda_0 t} \begin{pmatrix}
1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\
0 & 1 & t & \frac{t^2}{2!} \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Proof: \(A = \lambda_0 I + N\). Since \(IN = NI\), we have

\[
e^{tA} = e^{t\lambda_0} e^{tN} = e^{\lambda_0 t} e^{tN}.
\]

Note. If \(P(\lambda)\) is the characteristic polynomial of \(A\), then \(P(A) = 0\) (Cayley-Hamilton theorem). In particular, if \(P(\lambda) = (\lambda - \lambda_0)^n\), then \((A - \lambda_0 I)^n = 0\) and the computation of

\[
e^{tA} = e^{\lambda_0 t} e^{t(A - \lambda_0 I)}
\]

is very simple (similar to (b)).

(d) Complex eigenvalues:

\[
J = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \implies e^{\beta J} = \begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}.
\]

Proof: Since \(J^2 = -I\), \(J^3 = -J\), \(J^4 = I\), \ldots, we have

\[
e^{\beta J} = I + \beta J - \frac{\beta^2}{2!} I - \frac{\beta^3}{3!} J + \frac{\beta^4}{4!} I + \ldots = (\cos \beta)I + (\sin \beta)J.
\]

Corollary:

\[
A = \begin{pmatrix}
\alpha & -\omega \\
\omega & \alpha
\end{pmatrix} \implies e^{tA} = e^{\alpha t} \begin{pmatrix}
\cos \omega t & -\sin \omega t \\
\sin \omega t & \cos \omega t
\end{pmatrix}.
\]

Pf. \(A = \alpha I + \omega J\) and \(IJ = JI\).

(e) Linear change of variables.

If \(B = SAS^{-1}\), where \(S\) is an invertible \(n \times n\) matrix, then

\[
e^{tB} = Se^{tA}S^{-1}.
\]

Pf. \(B^2 = SA^2S^{-1}\), \(B^3 = SA^3S^{-1}\), \ldots

Remark. The last observation plus Examples (a)–(d) allow us to compute \(e^{tA}\) for all matrices: use \(S\) that puts the matrix \(A\) into its Jordan canonical form.
21. Linear equations with analytic coefficients


\[ y' + xy' = \nu y, \quad y(0) = 1, \quad (\nu \in \mathbb{R}). \]

Suppose that the solution \( y = y_\nu \) has a power series representation

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n. \]

Let us find the coefficients \( a_n \). We have

\[ y' = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n, \quad xy' = \sum_{n=0}^{\infty} na_n x^n, \]

and therefore

\[ (n+1)a_{n+1} + na_n = \nu a_n, \quad a_{n+1} = \frac{\nu - n}{n+1} a_n \]

for all \( n \geq 0 \). From the initial condition \( y(0) = 1 \) we find \( a_0 = 1 \), and so

\[ y_\nu(x) = 1 + \nu x + \frac{\nu(\nu - 1)}{2!} x^2 + \frac{\nu(\nu - 1)(\nu - 2)}{3!} x^3 + \ldots \]

If \( \nu \in \mathbb{N} \) then \( y_\nu \) is a polynomial, and if \( \nu = -1 \) then \( y_\nu \) is the sum of a geometric progression with radius of convergence equal to 1.

Note: \( y_\nu(x) = (1 + x)^\nu \), of course.

Exercise.

(a) Show that if \( \nu \notin \mathbb{N} \), then the radius of convergence is 1.

(b) Solve the following IVPs by the power series method:

\[ y' = y, \quad y(0) = 1; \quad y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 1. \]

We’ll be considering equations

\[ y'' + p(x)y' + q(x)y = 0 \quad (21.1) \]

such that both coefficients \( p \) and \( q \) are analytic in some neighborhood of \( x_0 \).

21.2. Power series solutions. Recall that analyticity of \( p(x) \) at \( x_0 \) means that there is a number \( R > 0 \) such that \( p(x) \) has a power series representation

\[ p(x) = \sum_{n} c_n(x-x_0)^n \quad \text{in} \quad |x-x_0| < R. \]

We must have \( R \leq \rho \), where \( \rho \) is the radius of convergence of the series,

\[ \rho = \lim \inf |c_n|^{-1/n}. \]

Note. Sometimes the easiest way to find the radius of convergence is to use the fact that \( \rho \) is the radius of the maximal disc in \( \mathbb{C} \) in which the function represented by the series is analytic. In other words, \( \rho \) is the distance from \( x_0 \) to the nearest singularity. For example, \( \rho = 1 \) for \( p(x) = (1 + x^2)^{-1} \); note that there are no singularities on \( \mathbb{R} \).
**Theorem.** Suppose that the coefficients of (21.1) have power series representations in the interval \( |x - x_0| < R \). Then every solution of the equation has a power series representation in this interval.

**Corollary.** If \( p \) and \( q \) are polynomials, then all solution of (21.1) are entire functions, (i.e. they have a power series representation on the whole line).

Idea of proof:

- Write \( y \) as a power series with undetermined coefficients.
- From the equation find a *recurrence relation* between the coefficients.
- This relation plus the initial conditions determine the coefficients uniquely.
- Check that the radius of convergence is a least \( R \).
- The sum of the power series automatically satisfies the equation.

21.3. **Example: Airy’s equation.**

\[
y'' - xy = 0.
\]

If

\[
y = \sum_{n=0}^{\infty} a_n x^n,
\]

then

\[
xy = \sum_{n=1}^{\infty} a_{n-1} x^n, \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.
\]

Equating the coefficients of \( x^n \), we get

\[
a_2 = 0, \quad (n+2)(n+1) a_{n+2} = a_{n-1} \quad (n \geq 1).
\]

It follows that

\[
a_2 = a_5 = \cdots = 0; \quad a_3 = \frac{a_0}{3 \cdot 2}, \quad a_6 = \frac{a_4}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, \quad \ldots
\]

\[
a_4 = \frac{a_1}{4 \cdot 3}, \quad a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}, \quad \ldots
\]

Clearly,

\[
|a_{3n}|, |a_{3n+1}| < \frac{C}{n!}
\]

so the radius of convergence is infinite and all solutions are entire functions. The coefficients are uniquely determined by the initial conditions

\[
a_0 = y_0, \quad a_1 = y_0' .
\]

Let \( A_1(x) \) be the solution of IVP(0; 1, 0),

\[
A_1(x) = 1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \ldots
\]
and let $A_2(x)$ be the solution of IVP($0; 0, 1$),

$$A_2(x) = x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \ldots$$

Then $A_1$ and $A_2$ are independent, and so the general solution of the Airy equation is

$$y(x) = C_1A_1(x) + C_2A_2(x).$$

Note. The functions $A_1(x)$, $A_2(x)$, or rather their certain linear combinations $Ai(x)$ and $Bi(x)$ known as the Airy functions, are examples of special functions. Airy functions can not be expressed in terms of elementary functions, but since they appear in many important applications, their properties have been extensively studied.

See Fig. 20.3 in the textbook for the graph of $A_1(x)$. Note the difference in the behavior for negative and positive $x$. The function oscillates on $\mathbb{R}_-$ but has a monotone, super-exponential growth on $\mathbb{R}_+$. This can be explained (informally) as follows: think of the Airy equation $y'' - xy = 0$ on the negative axis as a harmonic oscillator $y'' + \omega^2 y = 0$ whose frequency depends on $x$, $\omega^2 = -x$.

21.4. Aside: Riccati equation. Second order linear equations are closely related to the (first order, non-linear) Riccati equation

$$u' = a(x)u^2 + b(x)u + c(x).$$

(Riccati’s equation is linear if $a(x) = 0$ and Bernoulli if $c(x) = 0$.)

Lemma. If $y$ is a solutions of $y'' + py' + qy = 0$, then the function

$$u = \frac{y'}{y}$$

satisfies the first order equation

$$u' + pu + q + u^2 = 0.$$ 

Proof:

$$u' = \frac{y''y - y'y'}{y^2} = -py'y - qy^2 - y'y'. $$

Example. The Airy equation $y'' - xy = 0$ corresponds to Riccati’s equation

$$u' = x - u^2. \quad (21.2)$$

The general solution of (21.2) is therefore

$$u = \frac{C_1A'_1 + C_2A'_2}{C_1A_1 + C_2A_2};$$

it depends on just one parameter $C_2/C_1$. Earlier in the course we mentioned (21.2) as an example of a "non-solvable" (in elementary functions) equation.
22. Power series method for equations with meromorphic coefficients

We will now consider equations
\[ P(x)y'' + Q(x)y' + R(x)y = 0 \]
such that the coefficients \( P, Q, R \) are analytic at \( x_0 \). We can rewrite the equations in the normal form,
\[ y'' + p(x)y' + q(x)y = 0, \]
but the coefficients \( p, q \) will be meromorphic functions (the ratios of two analytic functions).

If \( P(x_0) \neq 0 \), then the functions \( p \) and \( q \) are analytic at \( x_0 \) and we say that \( x_0 \) is an ordinary point. If \( P(x_0) = 0 \), then \( x_0 \) is a singular point. In this case, we should consider the equation on the intervals \( \{ x > x_0 \} \) and \( \{ x < x_0 \} \) separately.

22.1. Cauchy-Euler equation. This is a model example:
\[ x^2y'' + \alpha xy' + \beta y = 0, \]
or
\[ y'' + \frac{\alpha}{x}y' + \frac{\beta}{x^2}y = 0. \]

Note that in the last equation, the coefficient of \( y' \) has a pole of order (at most) 1, and the coefficient of \( y \) has a pole of order (at most) 2.

Consider the interval \( \mathbb{R}_+ = \{ x > 0 \} \), and make the following change of the independent variable:
\[ \log x = t, \quad x = e^t, \quad \mathbb{R}_+ \leftrightarrow \mathbb{R}, \quad 0 \leftrightarrow -\infty. \]

We have
\[
\begin{align*}
y' &= \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\dot{y}}{x}, \\
y'' &= (\frac{\dot{y}}{x})' = \frac{\ddot{y}x - \dot{y}\dot{x}}{x^2} = \frac{\ddot{y} - \dot{y}}{x^2}.
\end{align*}
\]

The new equation is linear with constant coefficients:
\[
(\ddot{y} - \dot{y}) + \alpha \dot{y} + \beta y = 0,
\]
or
\[
\ddot{y} + (\alpha - 1)\dot{y} + \beta y = 0.
\]

The eigenvalue equation is
\[ \lambda^2 + (\alpha - 1)\lambda + \beta = 0. \]

Traditional notation and terminology: \( \lambda_i = r_j \) are called the exponents at the singularity, and the equation
\[ F(r) = r(r-1) + \alpha r + \beta = 0 \]
is called the indicial equation.

(Shortcut: to derive the indicial equation, substitute \( y = x^r \) in the C.-E. equation; note \( e^{rt} = x^r \).)
The exponents at the singularity are
\[ r_{1,2} = \frac{1 - \alpha \pm \sqrt{(1 - \alpha)^2 - 4\beta}}{2} \]

**Theorem.** (i) If the exponents are real and distinct, then the general solution is
\[ y(t) = C_1 x^{r_1} + C_2 x^{r_2}, \]
(ii) If they are equal, then
\[ y(x) = C_1 x^r + C_2 x^r \log x. \]
(iii) If the exponents are complex \( \alpha \pm i\omega \), then
\[ y(x) = x^\alpha C_1 \cos(\omega \log x) + x^\alpha C_2 \sin(\omega \log x). \]

We can describe the behavior of solutions near the singular point. For example, in
the first case, if both exponents are positive, then all solutions tend to 0 at 0, if
both are negative, then all solutions are unbounded, and if \( r_1 > 0, r_2 < 0 \), then one
solution tends to zero, and the others are unbounded.

The case \( \{x < 0\} \) is similar: just write \(|x|\) instead of \( x \).

### 22.2. Frobenius theory.
Consider the equation
\[ y'' + p(x)y' + q(x)y = 0 \]
with meromorphic coefficients, and let \( x_0 \) be a singular point. The singular point
is called *regular* if the singularities of \( p \) and \( q \) at \( x_0 \) are not worse than those in the
CE equation, i.e. \( p \) has a pole of order at most 1, and \( q \) has a pole of order at most
2:
\[ p(x) = \frac{\alpha}{x - x_0} + \ldots, \quad q(x) = \frac{\beta}{(x - x_0)^2} + \text{const} \frac{1}{x - x_0} + \ldots, \]
where the dots denote analytic functions. Let \( r_j \) be the roots ("exponents") of the
indicial equation
\[ F(r) := r(r - 1) + \alpha r + \beta = 0. \]
We are interested in the behavior of the solutions near the singular point, and
it is natural to expect that it is similar to the behavior of the solutions of the
Corresponding CE equation. This turns out to be true except for the resonance
case
\[ r_1 - r_2 \in \mathbb{Z}. \]
Here is a precise statement. We will only describe the case of real exponents (the
complex case is similar and even simpler since there’s no resonance). Without loss
of generality, we will assume \( x_0 = 0 \).

**Theorem.** Suppose that the functions \( \tilde{p}(x) := xp(x) \) and \( \tilde{q}(x) := x^2 q(x) \) have a
power series representation on the interval \((-\rho, \rho)\), and suppose that the exponents
\( r_j \) are real. Then the following is true in each of the two intervals \((-\rho, 0)\) and \((0, \rho)\).
(i) If \( r_1 - r_2 \notin \mathbb{Z} \), then there are two (independent) solutions of the form
\[ y_j = |x|^{r_j}(1 + \ldots), \]
where the dots stand for a power series which converges on \((-\rho, \rho)\) and is zero at
zero.
(ii) If \( 0 \leq r_1 - r_2 \in \mathbb{Z} \), then there is a solution \( y_1 \) of the form as above.
22.3. Remarks. (a) The proof (and the use of the theorem) is similar to the analytic case: we introduce undetermined coefficients and find a recurrence relation. In the equation
\[ x^2 y'' + \tilde{p}(x)xy' + \tilde{q}(x)y = 0, \quad (x > 0), \]
we set
\[ y(x) = \sum_{n \geq 0} a_n x^{r+n}, \quad \tilde{p}(x) = \sum p_n x^n, \quad \tilde{q}(x) = \sum q_n x^n. \]
Since
\[ xy' = \sum_{n \geq 0} (r+n)a_n x^{r+n} \]
\[ x^2 y'' = \sum_{n \geq 0} (r+n)(r+n-1)a_n x^{r+n} \]
\[ \tilde{p}(x)xy' = \sum_{n \geq 0} \left( \sum_{k+l=n} (r+l)ap_k \right) x^{r+n} \]
\[ \tilde{q}(x)y = \sum_{n \geq 0} \left( \sum_{k+l=n} a_lq_k \right) x^{r+n} \]
we get the equation
\[ (r+n)(r+n-1)a_n + \sum_{k+l=n} [(r+l)p_k + q_k]a_l = 0, \]
i.e.
\[ [(r+n)(r+n-1) + (r+n)p_0 + q_0]a_n = -\sum_{l=0}^{n-1} [(r+l)p_{n-l} + q_{n-l}]a_l. \]
Note that
\[ (r+n)(r+n-1) + (r+n)p_0 + q_0 = F(r+n). \]
In particular, for \( n = 0 \) the recurrence relation gives
\[ F(r)a_0 = 0. \]
We see that the condition \( F(r) = 0 \) is necessary for the existence of a non-trivial solution of the form \( y = x^r \cdot \text{(analytic function)}. \) This condition is also sufficient – we only need to note that if \( r \) is the largest exponent, then \( F(r+n) \neq 0 \) for \( n \geq 1 \), and the same is true for the smallest exponent in the non-resonance case. (We also need to check the convergence of the series.)

(b) In the resonance case, we can find the second (independent) solution using the Wronskian formula:
\[ y_2 = y_1 \int \frac{W}{y_1^2}, \quad W = e^{-\int p}. \]
One can show that if \( r_1 = r_2 \), then \( y_2 \) has a logarithmic term (as in the CE equation), but if \( 0 < r_1 - r_2 \in \mathbb{Z} \), then we may or may not have a logarithmic term. In fact, it is known that in the resonance case, there is a second solution of the form
\[ y_2(x) = \text{const} \ y_1(x) \log |x| + |x|^{r_2}(1 + \ldots), \]
where the constant can be zero (or non-zero) if \( r_1 \neq r_2 \).
23. Hypergeometric functions

23.1. Infinity as a singular point. In the case of rational coefficients, we can think of infinity as a singular point. The change of variables $t = x^{-1}$ transforms $x_0 = \infty$ to $t_0 = 0$. We have

$$y' = \frac{\dot{y}}{x} = -t^2 \dot{y},$$

and

$$y'' = -\frac{2t}{x} y - t^2 \frac{\ddot{y}}{x} = 2t^3 \dot{y} + t^4 \ddot{y},$$

so the new equation is

$$\ddot{y} + P(t) \dot{y} + Q(t) y = 0$$

with

$$P(t) = \frac{2}{t} - \frac{1}{t^2} p \left( \frac{1}{t} \right), \quad Q(t) = \frac{1}{t^4} q \left( \frac{1}{t} \right).$$

In particular, $\infty$ is a regular singular point iff

$$p(x) = O \left( \frac{1}{x} \right), \quad q(x) = O \left( \frac{1}{x^2} \right) \text{ at } \infty.$$  

[Exercise: compute the exponents at infinity in the case of CE equation.]

23.2. Hypergeometric equation. Consider equations with rational coefficients such that all singular points (in $\hat{\mathbb{C}}$, including $\infty$) are regular. There are no such equations if the number of singular points is 0 or 1. If the number of singular points is 2, and the points are 0 and $\infty$, then the only such equation is CE. The first non-trivial case is when we have three regular singular points. Without loss of generality, the points are 0, 1, $\infty$ (use linear-fractional transformations).

**Theorem.** In this case, the equation is

$$x(1-x)y'' + [\gamma - (1 + \alpha + \beta)x]y' - \alpha \beta y = 0$$

for some values of the parameters $\alpha, \beta, \gamma$.

The singularities are:

- $x = 0, \quad r_1 = 0, \quad r_2 = 1 - \gamma$,
- $x = 1, \quad r_1 = 0, \quad r_2 = \gamma - \alpha - \beta$,
- $x = \infty, \quad r_1 = \alpha, \quad r_2 = \beta$.

Consider the singular point $x_0 = 0$. If $\gamma \neq 0, -1, -2, \ldots$, then there is an analytic solution (Frobenius series corresponding to $r_1 = 0$)

$$y = \sum_{n \geq 0} a_n x^n$$

The recurrence relation is

$$a_{n+1} = \frac{(\alpha + n)(\beta + n)}{(n + 1)(n + \gamma)} a_n.$$
Choosing \(a_0 = 1\), we get the solution

\[
F(\alpha, \beta, \gamma; x) := 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha \cdot (\alpha + 1) \cdot \beta \cdot (\beta + 1)}{1 \cdot 2 \cdot \gamma \cdot (\gamma + 1)} x^2 + \ldots
\]

This is called hypergeometric series. The series converges in \((-1, 1)\). If we denote \((a)_n := a(a + 1) \ldots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}\), then

\[
F(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} x^n.
\]

23.3. **Special cases.** (i) If \(\beta = -N \in -\mathbb{N}\), then \(a_{N+1} = 0\) and the function is a polynomial of degree \(N\). These polynomials are related to the so called Legendre polynomials \(P_N\), which are defined by the equation

\[
(1 - x^2) y'' - 2xy' + \alpha(\alpha + 1)y = 0.
\]

Here the singular points are \(\pm 1, \infty\). If we change the variable \(x \mapsto (1 - x)/2\), then we find

\[P_N(x) = \text{const} \cdot F(N + 1, -N, 1, (1 - x)/2).\]

The case \(\alpha = -N\) is similar. Exercise: relate to Chebychev’s polynomials defined by the equation

\[
(1 - x^2)y'' - xy' + \alpha^2 = 0.
\]

(ii) There are other elementary hypergeometric functions. For example,

\[
F(1, 1, 2; x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots = -1 + \frac{\log 1}{1 - x},
\]

which follows from the relation

\[a_{n+1} = \frac{n + 1}{n + 2} a_n.\]

Another example is

\[
F(\alpha, \beta, \alpha; x) = (1 - x)^{-\beta}, \quad a_{n+1} = \frac{\beta + n}{n + 1}.
\]

We also have

\[
F(1/2, 1/2, 3/2; x^2) = \frac{\arcsin x}{x},
\]

\[
F(1/2, 1/2, 3/2; x^2) = \frac{\arctan x}{x}.
\]

(Hint: differentiate the arc-functions.)

(iii) The above examples are exceptional. A typical hypergeometric function is not elementary. An important example is the elliptic integral

\[
\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k(\sin \phi)^2}} = F(1/2, 1/2, 1; k).
\]
The Bessel equation with parameter $\nu \geq 0$ is
\[ x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad (x > 0). \]
The point $x_0 = 0$ is regular singular with exponents $\pm \nu$:
\[ r(r - 1) + r - \nu^2 = r^2 - \nu^2 = 0. \]

24.1. First solution. We always have the Frobenius solution with $r = \nu$,
\[ y_1(x) = x^\nu \sum_{n=0}^{\infty} a_n x^n. \]
The recurrence relation
\[ a_n = -\frac{a_{n-2}}{n(n + 2\nu)}, \quad (n \geq 2); \quad a_1 = 0, \]
shows that
\[ a_{odd} = 0, \]
and
\[ a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(1 + \nu)_m}, \]
where
\[ (1 + \nu)_m := (1 + \nu)(m + \nu)\ldots(1 + \nu + m - 1) = \frac{\Gamma(1 + \nu + m)}{\Gamma(1 + \nu)}. \]
Note that formula for $a_n$’s also makes sense for non-integer negative $\nu$’s, and the corresponding series satisfies the Bessel equation.

24.2. Definition. For $\nu \neq -1, -2, \ldots$, the Bessel function of order $\nu$ is
\[ J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(1 + \nu + m)} \left(\frac{x}{2}\right)^{2m+\nu}. \]
This function is the solution with
\[ a_0 = \frac{1}{2^\nu \Gamma(1 + \nu)}. \]
The power series in the definition of $J_\nu$ has infinite radius of convergence (why?), so $J_\nu$ extends to an analytic function in $\mathbb{C} \setminus \mathbb{R}_-$. If $\nu \in \mathbb{Z}_+$, then $J_\nu$ is an entire function:
\[ J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m + \nu)!} \left(\frac{z}{2}\right)^{2m+\nu}. \]
E.g., [D. Bernoulli 1732, Bessel 1824]
\[ J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma^2(m)} \left(\frac{z}{2}\right)^{2m}. \]
If $\nu \notin \mathbb{Z}$, then the functions $J_{\pm \nu}$ are independent solutions, so we have the following statement.

Theorem. If $\nu \notin \mathbb{Z}$, then the general solution of the Bessel equation of order $\nu$ is given by the formula
\[ y = C_1 J_\nu + C_2 J_{-\nu}. \]
(Note that the resonance case is \(2\nu \notin \mathbb{Z}\). The second solution has a logarithmic term iff \(\nu \in \mathbb{Z}\).)

**EX:** If \(\nu \geq 0\), then all solutions except for the multiples of \(J_\nu\) are unbounded at 0. Hint:

\[ y_2 = y_1 \int \frac{dx}{xy_1(x)}. \]

Several useful identities:

\[(xJ_1)' = xJ_0, \quad J_2 = \frac{2}{x}J_1 - J_0.\]

More generally,

\[J_{\nu+1} = \frac{2\nu}{x}J_\nu - J_{\nu-1}, \quad (\nu \geq 1).\]

\[J_{\nu+1} = -2J'_\nu + J_{\nu-1}, \quad (\nu \geq 1).\]

\[(x^\nu J_\nu)' = x^\nu J_{\nu-1}, \quad (\nu \geq 1).\]

\[(x^{-\nu} J_\nu)' = -x^{-\nu} J_{\nu+1}, \quad (\nu \geq 0).\]

### 24.3. Elementary Bessel’s functions.

**Theorem.** If \(\nu = 1/2 + n\) and \(n \in \mathbb{Z}\), then \(J_\nu\) is an elementary function. E.g.,

\[J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \quad (x > 0).\]

**Proof:** If \(\nu = 1/2\), then

\[a_n = -\frac{a_{n-2}}{n(n+1)}, \quad a_{2m} = \frac{(-1)^m a_0}{(2m+1)!},\]

and so

\[J_{1/2}(x) = \text{const} \sqrt{x} \sum_{m \geq 0} \frac{(-1)^m}{(2m+1)!} x^{2m} = \text{const} \frac{\sin x}{\sqrt{x}}.\]

To get the right constant use the value \(\Gamma(1/2) = \sqrt{\pi}\).

To get the expressions for \(J_{\pm 3/2}, J_{\pm 5/2}\) etc use the recurrence formula. \(\square\)

**Fact** (asymptotics at infinity):

\[J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos \left[ x - (2\nu + 1) \frac{\pi}{4} \right] + O(x^{-3/2}) \quad \text{as} \quad x \to \infty.\]

E.g.,

\[J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{4} \right), \quad J_{1/2}(x) \approx \sqrt{\frac{2}{\pi x}} \sin x, \quad J_1(x) \approx \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\pi}{4} \right).\]

An informal explanation of oscillations: almost constant frequency and negligible damping if \(|x| \gg 1|\).
24.4. Bessel’s functions on the imaginary axis. Modified Bessel’s functions \( I_\nu \) are defined as follows:

\[
J_\nu(ix) = i^\nu I_\nu(x), \quad I_\nu(x) := x^\nu \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m}, \quad (x > 0).
\]

Note that all coefficients are positive, so \( I_\nu(x) \to \infty \) as \( x \to \infty \), no oscillations. For example,

\[
I_{1/2} = \sqrt{\frac{2}{\pi x}} \sinh x, \quad I_{1/2} = \sqrt{\frac{2}{\pi x}} \cosh x.
\]

**Theorem.** \( I_\nu \) satisfies the “modified” Bessel’s equation

\[
x^2 y'' + xy' - (x^2 + \nu^2)y = 0.
\]

**Proof:** The expressions \( x^2 (d^2y/dx^2) \) and \( x(dy/dx) \) don’t change if \( x \mapsto ix \) but \( x^2 y \) becomes \(-x^2 y\). \( \square \)

If \( \nu \notin \mathbb{Z} \), then the general solution of the modified equation is \( y = C_1 I_\nu + C_2 I_{-\nu} \).

24.5. Change of variables. Many equations can be solved in terms of Bessel’s functions. Just two examples:

**Lemma.** Suppose \( w(z) \) is a solution of the Bessel equation with parameter \( \nu \). Then the function

\[
y(x) = x^\alpha w(kx^\beta)
\]

satisfies the equation

\[
x^2 y'' + Axy' + (B + Cx^{2\beta})y = 0
\]

with

\[
A = 1 - 2\alpha, \quad B = \alpha^2 - \beta^2 \nu^2, \quad C = \beta^2 k^2.
\]

**Example.** Airy’s equation is \( y'' = xy \), or

\[
x^2 y'' - x^3 y = 0
\]

We have

\[
A = 0, \quad B = 0, \quad C = -1, \quad \beta = \frac{3}{2},
\]

so

\[
\alpha = \frac{1}{2}, \quad \nu = \frac{1}{3}, \quad k = \frac{2}{3},
\]

and the general solution is

\[
y = C_1 \sqrt{\pi} I_{1/3} \left(\frac{2}{3} x^{3/2}\right) + C_2 \sqrt{\pi} I_{-1/3} \left(\frac{2}{3} x^{3/2}\right).
\]

**Example.** ("Aging spring")

\[
\ddot{y} + \omega^2 e^{-\varepsilon t} y = 0,
\]

where \( 0 < \varepsilon \ll 1 \). Change: \( x = ae^{bt} \), the constants to be chosen later. We have

\[
\dot{y} = y' \dot{x} = y' b x, \quad \ddot{y} = y'' (bx)^2 + y' b^2 x.
\]

Since

\[
e^{-\varepsilon t} = (x/a)^{-\varepsilon/b},
\]
it follows that if we choose \( b = -\varepsilon/2 \),
then the new equation is:

\[
x^2y'' + xy' + \frac{\omega^2}{a^2b^2}x^2y = 0.
\]

Finally, we set

\[
a = -\omega/b = (2\omega)/\varepsilon,
\]
so the equation is Bessel of order 0. Its general solution is

\[
y(t) = C_1 J_0(x) + C_2 Y_0(x), \quad x = \frac{2\omega}{\varepsilon}e^{-(\varepsilon t)/2},
\]
where \( Y_0 \) is the second independent solution.

**Exercise** (Riccati equations). We already mentioned that the substitution \( u = y'/y \)
transforms \( y'' + py' + qy = 0 \) into

\[
u' + pu + q + u^2 = 0.
\]
Similarly, the substitution \( v = -y'/y \) transforms \( y'' + py' + qy = 0 \) into

\[
v' = pv + q + v^2.
\]
Solve the following equations in terms of Bessel’s functions:

\[
u' = x^2 - u^2, \quad u' = x - u^2
\]
and

\[
v' = v^2 \pm x^2, \quad v' = v^2 \pm x.
\]