PROBLEM SET 5
Due on Monday, November 9

I. Oscillations
1) Problem 15.4
2) Show that the system in Problem 26.1(v) has a unique periodic solution. Is this solution stable?

III. Linear systems with constant coefficients
3) Problem 29.3(i)
4) Compute the matrix-valued function $e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$ for $A = \begin{pmatrix} 7 & -4 \\ 1 & 3 \end{pmatrix}$. Solve the IVP $\dot{x} = Ax$, $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
5) Rewrite the $n$-th order linear equation with constant coefficients $a_0 x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0$ as a linear system $\dot{y} = Ay$ in $\mathbb{R}^n$. Show that the eigenvalues of the matrix $A$ are exactly the roots of the polynomial $p(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$.

II. Power series solutions
6) Problem 20.3
7-8) Find a series solution in powers of $x$ of the IVP $y'' + \lambda xy = 0$, $y(0) = 1$, $y'(0) = 0$, where $\lambda$ is a given positive parameter.
Approximating the solution by the Taylor polynomial of degree 6, find the minimal number $L > 0$ such that the boundary value problem $y'' + \lambda xy = 0$, $y'(0) = 0$, $y(L) = 0$ has a non-trivial solution.
[Interpretation: when a uniform vertical column will buckle under its own weight? The unknown function $y(x)$ is the angle of deflection from the vertical direction; $x = 0$ is the free top end and $x = L$ is the fixed bottom end of the column; the column parameter $\lambda$ depends on the density and the Young’s modulus of the material, and on the shape of the cross section of the column. The column can buckle if there is a non-trivial solution to the BVP.]