Problem 1 10 pts  

(i) Write down the general solution to the equation $\dot{x} = A x$.

(ii) After changing to a coordinate system referred to the eigenvectors the equation will become

$$\frac{d\tilde{x}}{dt} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tilde{x}.$$  

i.e.,

$$\frac{d\tilde{x}}{dt} = 0 \quad \text{and} \quad \frac{d\tilde{y}}{dt} = \lambda_2 \tilde{y}.$$  

By solving these equations draw the phase portrait for the $(\tilde{x}, \tilde{y})$ system, and hence sketch the phase portrait for the original coordinates.

(iii) Draw the phase portrait for the equation

$$\frac{dx}{dt} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} x.$$  

Solution. (i) As in (28.1) in the book the general solution is

$$x(t) = Ae^{\lambda_1 t} v_1 + Be^{\lambda_2 t} v_2 = A v_1 + B e^{\lambda_2 t} v_2$$

(ii) The solution to the equation in the $(\tilde{x}, \tilde{y})$ system is $\tilde{x} = \tilde{x}(0)$ and $\tilde{y} = \tilde{y}(0) e^{\lambda_2 t}$. So the phase portrait in the $(\tilde{x}, \tilde{y})$ coordinates will be straight lines parallel to the $\tilde{y}$-axis going towards or away from the line $\tilde{y} = 0$ according to the sign of $\lambda_2$. Translating this back to the original coordinates we see that the phase portrait consists of lines parallel to $v_2$ going towards or away from the line spanned by $v_1$ according to the sign of $\lambda_2$.

(iii) The matrix has characteristic polynomial $\lambda^2 + 3\lambda = \lambda(\lambda + 3)$, so it has eigenvalues 0 and $-3$. The eigenvector corresponding to 0 is $(1, 1)^T$, and the eigenvector corresponding to $-3$ is $(2, -1)^T$. The phase portrait should consist of a collection of lines parallel to $(2, -1)^T$, with arrows pointing toward the line spanned by $(1, 1)^T$. \qed

Problems 2-3 20 pts  

(Problem 31.1 (i, ii, iv, v, vi, ix))

Draw the phase portrait of the equation $dx/dt = Ax$ when the eigenvalues and eigenvectors of $A$ are the following:

(i) $\lambda_1 = 3$ with $v_1 = (1, 1)$ and $\lambda_2 = -2$ with $v_2 = (1, -2)$;

(ii) complex conjugate eigenvalues $\lambda_{\pm} = -1 \pm 3i$, with $\dot{x} < 0$ when $x = 0$ and $y > 0$;

(iv) $\lambda_1 = -2$ with $v_1 = (2, 1)$ and $\lambda_2 = -3$ with $v_2 = (1, -1)$;

(v) a single eigenvalue $\lambda = -3$ with eigenvector $(1, -1)$ and $\dot{x} > 0$ when $x = 0$ and $y > 0$;

(vi) $\lambda = \pm 2i$, where $\dot{y} < 0$ when $y = 0$ and $x > 0$;

(ix) a single eigenvalue $\lambda = -7$, with the matrix $A$ a multiple of the identity

Solution. (i) When $\lambda_1 = 3$ with $v_1 = (1, 1)$ and $\lambda_2 = -2$ with $v_2 = (1, -2)$, the two eigenvalues are both real but with opposite sign. The origin is therefore a saddle point, attracting along the the direction $(1, -2)$ and repelling along the direction $(1, 1)$.

(ii) When there are complex conjugate eigenvalues $\lambda_{\pm} = -1 \pm 3i$, with $\dot{x} < 0$ when $x = 0$ and $y > 0$, the origin is a stable spiral (because the real part of $\lambda_{\pm}$ is negative) with counterclockwise trajectories (because of the condition on $\dot{x}$).

(iv) When $\lambda_1 = -2$ with $v_1 = (2, 1)$ and $\lambda_2 = -3$ with $v_2 = (1, -1)$, the origin is a stable node because both eigenvalues are negative. Also, trajectories are asymptotically parallel to $v_1$ because $-2 > -3$.

(v) When there is a single eigenvalue $\lambda = -3$ with eigenvector $(1, -1)$ and $\dot{x} > 0$ when $x = 0$ and
\( y > 0 \), the origin is a stable improper node because \(-3 < 0\).

(vi) When \( \lambda = \pm 2i \) where \( \dot{y} < 0 \) when \( y = 0 \) and \( x > 0 \), the origin is a center because the eigenvalues are complex conjugates with real part zero. Trajectories are clockwise because of the condition on \( \dot{y} \).

(ix) When there is a single eigenvalue \( \lambda = -7 \) with the matrix \( A \) a multiple of the identity, the origin is a stable star because every nonzero vector is an eigenvector. \( \square \)

**Problem 4**

-10 pts (Problem 32.1) The Hartman-Grobman theorem guarantees that the phase portrait for a nonlinear equation looks like the linearised phase portrait sufficiently close to a stationary point *provided that the eigenvalues have non-zero real part*. In particular, the linearised system may not give a qualitatively correct picture when the linearised equation produces a center, as this example demonstrates.

First show that the origin is a center for the linearised version of the equation

\[
\begin{align*}
\dot{x} &= -y + \lambda(x^2 + y^2) \\
\dot{y} &= x + \lambda y(x^2 + y^2)
\end{align*}
\]

Now write down the equation satisfied by \( r \), where \( r^2 = x^2 + y^2 \), and hence show that the stability of the origin depends on the sign of \( \lambda \). Draw the phase portrait for \( \lambda < 0 \).

**Solution.** The Jacobian matrix is

\[
\begin{pmatrix}
3x^2 + y^2 & -1 + 2xy \\
1 + 2xy & x^2 + 3y^2
\end{pmatrix}
\]

Which at \((0, 0)\) is

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

, which has characteristic polynomial \( \lambda^2 + 1 \) and thus has eigenvalues \( \pm i \). This implies that the origin is a center for the linearised system. At \((1, 0)^T\), we check that the derivative is \((0, 1)^T\), so the center rotates counterclockwise.

For the second part of the problem, we can implicitly differentiate the equation with respect to \( t \), giving:

\[
2r \dot{r} = 2x \dot{x} + 2y \dot{y}
= 2x(-y + \lambda x(x^2 + y^2)) + 2y(x + \lambda y(x^2 + y^2))
= 2\lambda(x^2 + y^2)^2
= 2\lambda r^4
\]

Which means \( \dot{r} = \lambda r^3 \). Thus if \( \lambda > 0 \), solutions spiral away from the origin (\( r \) is always strictly increasing), and if \( \lambda < 0 \), solution spiral towards the origin (\( r \) is always strictly decreasing). The phase portrait for \( \lambda < 0 \) should have counterclockwise spirals going toward the origin. \( \square \)

**Problem 5**

-10 pts (Problem 33.1(ii)) For the following model, describe first the type of situation being modeled, then find the stationary points, determine their stability type and draw the phase portrait for \( x, y \geq 0 \). Finally, say what the phase portrait means for the two species.

\[
\begin{align*}
\dot{x} &= x(2 - x - y) \\
\dot{y} &= y(2 - x/4 - 2y)
\end{align*}
\]

and find the equations of the curves along which the solutions move.

**Solution.** This is a competing species situation. We find the stationary points by setting \( x(2 - x - y) \) and \( y(2 - x/4 - 2y) \) to be 0. We thus get the following stationary points: \((0, 0)\), \((0, 1)\), \((2, 0)\) and \((8/7, 6/7)\).

We now want to look at the linearization about the four points above. The Jacobian matrix is given by

\[
\begin{pmatrix}
2 & -2x - y \\
-y/4 & 2 - x/4 - 4y
\end{pmatrix}
\]

Near the origin, the Jacobian evaluates to

\[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]
with a repeated real eigenvalue 2. Since all vectors are eigenvectors and the eigenvalue is positive, the stability type of $(0, 0)$ is that of a stable star. Thus the linearized phase portrait near the origin is a star with lines marked with arrows pointing outwards.

At $(0, 1)$ the matrix evaluates to
\[
\begin{pmatrix} 1 & 0 \\ -1/4 & -2 \end{pmatrix},
\]
which has eigenvalues $-2$ and $1$ with respective eigenvectors $(0, 1)^T$, $(-12, 1)^T$. Thus $(0, 1)$ is a saddle point and the linearized phase portrait near $(0, 1)$ has unstable direction along the vector $(-12, 1)^T$ and stable direction along the $y$-axis.

At $(2, 0)$ the matrix evaluates to
\[
\begin{pmatrix} -2 & -2 \\ 0 & 3/2 \end{pmatrix},
\]
which has eigenvalues $-2$ and $3/2$ with respective eigenvectors $(1, 0)^T$, $(-4/7, 1)^T$. Thus $(2, 0)$ is a saddle point and the linearized phase portrait near it has unstable direction along the vector $(-4/7, 1)^T$ and stable direction along the $x$-axis.

At $(8/7, 6/7)$ the matrix evaluates to
\[
\begin{pmatrix} -8/7 & -8/7 \\ -3/14 & -12/7 \end{pmatrix},
\]
which has eigenvalues $-2$ and $-6/7$ with respective eigenvectors $(4/3, 1)^T$, $(-4, 1)^T$. Thus $(8/7, 6/7)$ is a stable node.

Since all eigenvalues have real part non-zero, Hartman-Grobman tells us that the phase portrait of the original nonlinear problem can be approximated by the linearized phase portraits when we’re close to the stationary points. Thus, joining up the local phase portraits gives us the global picture.

Finally, notice that the interior stationary stable node $(8/7, 6/7)$ attracts all trajectories, so any initial condition that includes some of both species will lead to a state of coexistence where the ratio of species $x$ with respect to species $y$ is $4/3$. As a result, this competitive model is a situation of coexistence. If there is only species $x$, then it will settle to the equilibrium $x = 2$, while if there is only species $y$, it will settle to its own equilibrium $y = 1$.

\[\square\]

**Problem 6**

10 pts (Problem 33.1(iv)) For the following model, describe first the type of situation being modeled, then find the stationary points, determine their stability type and draw the phase portrait for $x,y \geq 0$. Finally, say what the phase portrait means for the two species.

\[
\dot{x} = x(1 - 2y) \\
\dot{y} = y(-2 + 3x)
\]

and find the equations of the curves along which the solutions move.

**Solution.** This is a predator-prey situation. The stationary points are $(0, 0)$ and $(2/3, 1/2)$. The matrix of partial derivatives is
\[
\begin{pmatrix} 1 - 2y & -2x \\ 3y & -2 + 3x \end{pmatrix}.
\]

At the origin this is
\[
\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},
\]
with eigenvalues 1 and $-2$ so the origin is a saddle point with unstable direction equal to the $x$-axis and stable direction equal to the $y$-axis. At $(2/3, 1/2)$ this is
\[
\begin{pmatrix} 0 & -4/3 \\ 3/2 & 0 \end{pmatrix},
\]
which has eigenvalues $\pm i\sqrt{2}$. Thus, the linearization has a centre and the orbits are circling counterclockwise. However, Hartman-Grobman does not apply so we don’t immediately know if the original system has a centre or a spiral at $(2/3, 1/2)$. 

\[\square\]
To resolve this, we divide $\dot{y}$ by $\dot{x}$ to discover

$$\frac{dy}{dx} = \frac{y(-2 + 3x)}{x(1 - 2y)}.$$ 

Separating gives

$$\left( \frac{1}{y} - 2 \right) dy = \left( -\frac{2}{x} + 3 \right) dx,$$

and integrating gives

$$\log(y) - 2y = -2 \log(x) + 3x + C.$$ 

Hence the equations for the curves along which the solutions move are

$$ye^{-2y} = Kx^{-2}e^{3x}.$$ 

Distinct trajectories will correspond to distinct choices of $K$. Notice that

$$\lim_{y \to \infty} ye^{-2y} = 0.$$ 

Therefore, for any fixed values of $K \neq 0$ and $x \neq 0$, there are at most finitely many values of $y$ that solve the above equation. If the original system has a spiral (and not a cycle), then along any trajectory near the stationary point (i.e. for any fixed value of $K$) there will be infinitely many points of intersection between the trajectory and the vertical line $x = 2/3$. This is a contradiction by what we have just seen. Therefore, it must be that the original system also has a centre at the stationary point $(2/3, 1/2)$. Overall, the system is oscillatory.

\[\square\]

**Problem 7-10 pts** (Problem 33.1(vii)) For the following model, describe first the type of situation being modeled, then find the stationary points, determine their stability type and draw the phase portrait for $x, y \geq 0$. Finally, say what the phase portrait means for the two species.

\[
\begin{align*}
\dot{x} &= x(3 - x - y) \\
\dot{y} &= y(-2 + x).
\end{align*}
\]

**Solution.** This is a predator-prey situation. The stationary points are $(0, 0)$, $(3, 0)$, and $(2, 1)$. The matrix of partial derivatives is

$$\begin{pmatrix} 3 - 2x - y & -x \\ y & -2 + x \end{pmatrix}.$$ 

At the origin it is

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix},$$

so $(0, 0)$ is a saddle point. At $(3, 0)$ is is

$$\begin{pmatrix} -3 & -3 \\ 0 & 1 \end{pmatrix},$$

so the eigenvalues are $-3$ and $1$ with eigenvectors $(1, 0)$ and $(-3, 4)$ respectively. Thus $(3, 0)$ is also a saddle point. At $(2, 1)$ we get

$$\begin{pmatrix} -2 & -2 \\ 1 & 0 \end{pmatrix},$$

so the eigenvalues are $-1 + i$ and $-1 - i$. Thus, $(2, 1)$ is a stable spiral. Furthermore, checking at some nearby point reveals that the rotation is counterclockwise. Overall, we see that if both $x$ and $y$ are initially positive then the system will approach the stationary point at $(2, 1)$. \[\square\]

**(X1) 10 points** Sketch a phase diagram of the system

$$\dot{z} = -z^2, \quad z(t) \in \mathbb{C}$$
Solution 1. First, \( z = x + iy \), then the system is equivalent to
\[
\begin{align*}
\dot{x} &= y^2 - x^2 \\
\dot{y} &= 2xy
\end{align*}
\]
Considering the lines \( x = 0, \ y = 0, \ x = \pm \sqrt{3}y \) we can get a sketch of the whole phase portrait:

Solution 2. Using substitution \( s(t) = \int |z(t)|^4 \, dt \), we have \( ds = |z|^4 \, dt \) and so
\[
\begin{align*}
\frac{dz}{dt} &= -z^2 = \frac{|z|^4}{z^2} \\
\frac{dz}{ds} &= \frac{dz}{dt} \frac{dt}{ds} = -\frac{1}{z^2}
\end{align*}
\]
Then integrating both sides we get \( z^3 = 3s + a + ib \). Let \( z = x + iy \), \( z^3 = x^3 - 3xy^2 + i(3x^2y - y^3) \) hence taking the imaginary part of both sides, we get
\[
3x^2y - y^3 = b
\]
When \( b = 0 \), we have the orbits given by \( y = 0, \ y = \pm \sqrt{3}x \). Choosing different nonzero \( b \)'s then we can get the phase portrait.

Solution 3. Using substitution \( w = z^3 \), we have
\[
\dot{w} = 3z^2 \dot{z} = 3z^2(-z^2) = -3|z|^4 = -3|w|^{4/3}
\]
which has a simple phase diagram with left-pointing arrows at each point. Now since \( w = z^3 \), each point of \( w \) corresponds to three points of \( z \) at an angle of \( 2\pi/3 \) apart, and since
\[
\frac{\dot{w}}{w} = \frac{3z^2 \dot{z}}{z^3} = \frac{\dot{z}}{z},
\]
we know that the angle of \( \dot{z} \) relative to \( z \) is the same as that of \( \dot{w} \) relative to \( w \), from which we can deduce the phase diagram for \( z \) from the one for \( w \).